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A NEW RUNGE-KUTTA METHOD FOR SOLUTION
OF THE POWER SYSTEM SWING EQUATION

by

John David Duffin

A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled A New Runge Kutta Method for Solution of the Power System Swing Equation submitted by John David Duffin in partial fulfillment of the requirements for the degree of Master of Science.



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ABSTRACT

Maintenance of continuous electrical service by a power company is possible only if its power system is stable when subjected to any disturbances which change power flow. Extensive study is necessary to guarantee this continuity of supply. The study is termed power system stability analysis. The analysis, in turn, is wholly dependent on a non-linear differential equation - the swing equation - which describes the motion of each synchronous generator on a power system.

The advent of the digital computer has enabled power systems engineers to adopt new and improved methods of system stability analysis. Runge-Kutta integration methods have replaced the step by step methods previously used to solve the swing equation numerically.

Two Runge-Kutta methods are in popular use in this application:

- 1) the method of Johnson and Ward⁽²⁾;
- 2) the method of Gill⁽¹⁰⁾.

The intent of the writer is to analyze these and various other integration techniques with a view to finding a procedure which gives optimum accuracy of solution of the swing equation, with minimal computer computation time. The writer has derived a new Runge-Kutta scheme to achieve that objective, and this method is discussed in this thesis.

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TABLE OF CONTENTS

	<u>Page</u>
TABLE OF CONTENTS	i
LIST OF FIGURES	ii
I. INTRODUCTION	1
II. A DERIVATION OF RUNGE-KUTTA METHODS FOR INTEGRATION OF A FIRST-ORDER DIFFERENTIAL EQUATION	4
III. INTEGRATION OF THE SWING EQUATION USING FIRST- ORDER DIFFERENTIAL EQUATION TECHNIQUES	12
3.1 Conversion of Four Basic Runge-Kutta Processes	19
3.2 Solution of a Sample Problem	24
3.3 Application of Merson's Method	30
IV. A DERIVATION OF RUNGE-KUTTA METHODS FOR INTEGRATION OF A SECOND-ORDER DIFFERENTIAL EQUATION	40
V. SELECTION OF THE METHOD MOST SUITABLE FOR DIGITAL COMPUTER INTEGRATION OF THE SWING EQUATION	52
5.1 Development of the Johnson-Ward Formulae	52
5.2 Formulation of New Procedures	60
VI. INTERVAL-CHANGING PROCEDURES FOR INTEGRATION OF THE SWING EQUATION	76
VII. SUMMARY	87
APPENDIX A - APPLICATION OF DE VOGELAERE'S METHOD	A-1
APPENDIX B - METHOD OF APPLICATION OF RUNGE-KUTTA TECHNIQUES TO A MULTIMACHINE SYSTEM	B-1
APPENDIX C - SOME COMPUTER RESULTS	C-1
BIBLIOGRAPHY	

LIST OF FIGURES

<u>Figure</u>		<u>Page</u>
3.1	Single Line Diagram of a Power System	25
3.2	Accuracy Comparison Flow Chart	27
3.3	Swing Curves - Example 3.1	29
3.4	Swing Curves - Example 3.2	36
3.5	Truncation Error Estimate - Example 3.2	38
5.1	Swing Curves - Example 5.1	59
5.2	Swing Curves - Example 5.2	73
5.3	Swing Curves - Example 5.2	75
6.1	Swing Curves - Example 6.1	86
7.1	Solution of Example 7.2, Using Eqn. (5.17)	91
7.2	Solution of Example 7.2, Using Eqn. (5.46)	92
7.3	Solution of Example 7.2, Using Eqn. (3.23)	93
7.4	Solution of Example 7.2, Using Eqn. (A.1 - A.5)	94
7.5	Solution of Example 7.2, Using Eqn. (5.12)	95
A.1	Use of De Vogelaere's Procedure	A-2

I. INTRODUCTION

1.1 The Swing Equation and Its Solution

The swing equation is a non-linear, second-order differential equation derived from the fundamental laws of mechanics⁽¹⁾. It describes the motion of each synchronous machine on a power system. Parameters needed for a description of the basic form of the equation are as follows:

P_a is accelerating power, the difference between net mechanical input P_i and electrical power output P_u , both corrected for losses;

M is the "inertia constant" of a machine, a term proportional to one representing kinetic energy at rated speed; and

δ is the displacement angle of the machine rotor with respect to an imaginary reference axis rotating at synchronous speed; and

t is the elapsed time measured from the beginning of a disturbance.

The swing equation is of the form

$$M \frac{d^2 \delta}{dt^2} = P_a = P_i - P_u$$

Mechanical input power is usually assumed constant, because of the relative slowness of governor action. In the most general case, assuming no damping and constant internal voltage,

$$P_a = f(\delta).$$

If damping and field decrement are taken into account, the swing equation takes the form

$$M \frac{d^2 \delta}{dt^2} = P_a = f(t, \delta, \frac{d\delta}{dt}).$$

In power systems design, swing curves, loci of δ vs. time governed by the swing equation, are calculated to evaluate system response following an assumed disturbance in the generator power outputs. An analytical method of solving the swing equation has not been found to date. The most popular procedure for obtaining the swing curves has been to obtain approximate solutions of the swing equation by numerical integration.

The most important integration methods used have been the following:

- a) A step-by-step method generally used in conjunction with the network analyzer⁽¹⁾ -
 - i) the acceleration, as calculated at the beginning of a particular time interval, remains constant from the centre of the preceding interval to the centre of the interval in question;
 - ii) the angular velocity, as computed at the centre of an interval, remains constant over that interval.
- b) The number series method, recently proposed by Rao and Rao⁽⁶⁾. This method is based on a closed-loop control system representation of the swing equation; the response is evaluated by trapezoidal integration of the convolution

integral.

- c) The Runge-Kutta fourth-order method and its variations.

The use of Runge-Kutta in this application was introduced by Johnson and Ward⁽²⁾.

The Johnson-Ward method was adjudged superior to the commonly-used step-by-step method by the following claims:

- a) it was more accurate than the priorly-used methods and took less computational time on a digital machine to accomplish this accuracy;
- b) it was self-starting - every integration step was begun anew without reliance on any values preceding those at the beginning of each iteration.

More recently^(3,4), another computational procedure has been used in place of the Johnson-Ward method. This is Gill's variation of Runge-Kutta - applied to the swing equation by using a substitution of variable which transformed the equation into two first-order equations.

Neither of these methods gives optimum accuracy and minimum computational time for stability studies on a digital computer. The writer derives and analyzes these and other methods, with a view to finding which is the most suitable in this regard.

II. DERIVATION OF RUNGE-KUTTA FOURTH-ORDER METHODS
TO SOLVE THE DIFFERENTIAL EQUATION

$$\frac{dy}{dx} = f(x, y); \quad y(x_0) = y_0$$

The derivation of first-order Runge-Kutta algorithms is certainly not new. It is presented here because a basic understanding of both the mechanics of the derivation, and the form of the notation used, is essential for a full comprehension of the sections following.

The general purpose of a Runge-Kutta method is to obtain an approximate numerical solution of a differential equation. In contrast to a formal Taylor Series solution,

$$\begin{aligned} y(x_0 + h) - y(x_0) &= h y'(x_0) + \frac{h^2}{2!} y''(x_0) \\ &+ \frac{h^3}{3!} y'''(x_0) + \dots \end{aligned} \quad (2.1)$$

which requires the evaluation of numerous derivatives at the point x_0 , the Runge-Kutta algorithm uses only one derivative of the function, and obtains $y(x_0 + h)$ at the expense of several evaluations of this derivative.

The aim of the derivation is to express $y(x_0 + h) - y(x_0)$ as a linear combination of values of the function between x_0 and $x_0 + h$. To this end we define $\{k_i\}$, $i = 1, 4$ such that

$$y(x_0 + h) - y(x_0) = \sum_{i=1}^4 \mu_i k_i + \text{truncation error} \quad (2.2)$$

where the $\{\mu_i\}$ are constants minimizing the truncation error.

In a fourth-order Runge-Kutta method, the term

$$\sum_{i=1}^4 \mu_i k_i$$

must correspond to the true Taylor expansion (2.1) in all terms up to and including those of order h^4 :

$$\begin{aligned} \sum_{i=1}^4 \mu_i k_i + O(h^5) &= h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y^{(3)}(x_0) \\ &+ \frac{h^4}{4!} y^{(4)}(x_0) + \dots \end{aligned} \quad (2.3)$$

To avoid the evaluation of explicit higher derivatives in (2.1), we set

$$k_1 = h \cdot f(x_0, y_0) \quad (2.4)$$

$$k_2 = h \cdot f(x_0 + \alpha h, y_0 + \beta k_1) \quad (2.5)$$

$$k_3 = h \cdot f(x_0 + \alpha_1 h, y_0 + \beta_1 k_1 + \gamma_1 k_2) \quad (2.6)$$

$$k_4 = h \cdot f(x_0 + \alpha_2 h, y_0 + \beta_2 k_1 + \gamma_2 k_2 + \delta_2 k_3) \quad (2.7)$$

The problem now is to find suitable values of the constants $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$, $\{\delta_i\}$ and $\{\mu_i\}$ such that (2.3) is satisfied.

Using the notation

$$f_x = \left[\frac{\partial f}{\partial x} \right]_{x=x_0}, \quad f_y = \left[\frac{\partial f}{\partial y} \right]_{x=x_0}, \quad f_{xyy} = \left[\frac{\partial^3 f}{\partial x \partial^2 y} \right]_{x=x_0}$$

etc.

$$Df = \frac{df}{dx} = \frac{\partial f}{\partial x} \frac{dx}{dx} + \frac{\partial f}{\partial y} \frac{dy}{dx} = f_x + f_y f_y$$

$$D^2 f = f_{xx} + 2ff_{xy} + f^2 f_{yy}$$

$$D^3 f = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f^3 f_{yyy}$$

$$D^4 f = f_{xxxx} + 4ff_{xxx} + 6f^2 f_{xxyy} + 4f^3 f_{xyyy} + f^4 f_{yyyy},$$

in lieu of (2.1) we may write

$$\begin{aligned} y(x_0 + h) - y(x_0) &= hf + \frac{h^2}{2!} \left(\frac{df}{dx} \right) + \frac{h^3}{3!} \left(\frac{d^2 f}{dx^2} \right) \\ &\quad + \frac{h^4}{4!} \left(\frac{d^3 f}{dx^3} \right) + O(h^5), \end{aligned} \quad (2.8)$$

Since

$$\frac{d^2 f}{dx^2} = D^2 f + f_y f_x + f f_y^2 = D^2 f + f_y Df$$

and

$$\frac{d^3 f}{dx^3} = D^3 f + f_y D^2 f + f_y^2 Df + 3Df Df_y$$

this becomes

$$\begin{aligned} y(x_0 + h) - y(x_0) &= hf + \frac{h^2}{2!} Df + \frac{h^3}{3!} (D^2 f + f_y Df) + \frac{h^4}{4!} (D^3 f \\ &\quad + f_y D^2 f + f_y^2 Df + 3Df Df_y) + O(h^5) \end{aligned} \quad (2.9)$$

In order to correlate (2.9) with (2.2) it is necessary to expand the expressions for the $\{k_i\}$ into their Taylor expansions about the point (x_0, y_0) .

Using the Taylor series for two variables, namely

$$\begin{aligned} f(x_0 + \alpha h, y_0 + \beta h) &= f(x_0, y_0) + \left[\alpha h \frac{\partial}{\partial x} + \beta h \frac{\partial}{\partial y} \right] f(x_0, y_0) \\ &+ \frac{1}{2!} \left[\alpha h \frac{\partial}{\partial x} + \beta h \frac{\partial}{\partial y} \right]^2 f(x_0, y_0) \\ &+ \dots \end{aligned} \quad (2.10)$$

we can obtain the following expressions for the k_i :

$$k_1 = hf. \quad (2.11)$$

$$\begin{aligned} k_2 &= hf + h^2(\alpha Df) + \frac{h^3}{2} (\alpha^2 D^2 f) + \frac{h^4}{6} (\alpha^3 D^3 f) \\ &+ O(h^5), \quad \text{if } \alpha = \beta. \end{aligned} \quad (2.12)$$

$$\begin{aligned} k_3 &= hf + h^2(\alpha_1 Df) + \frac{h^3}{2} (\alpha_1^2 D^2 f + 2\alpha_1 \gamma_1 f_y Df) \\ &+ \frac{h^4}{6} (\alpha_1^3 D^3 f + 3\alpha_1^2 \gamma_1 f_y D^2 f + 6\alpha_1 \gamma_1 Df Df_y) + O(h^5), \\ &\text{if } \alpha_1 = \beta_1 + \gamma_1. \end{aligned} \quad (2.13)$$

$$\begin{aligned}
 k_4 = & hf + h^2 \left[\alpha_2 Df \right] + \frac{h^3}{2} \left(\alpha_2^2 D^2 f + 2(\alpha \gamma_2 + \alpha_1 \delta_2) f_Y Df \right) \\
 & + \frac{h^4}{6} \left[\alpha_2^3 D^3 f + 3(\alpha^2 \gamma_2 + \alpha_1^2 \delta_2) f_Y D^2 f + 6\alpha \gamma_1 \delta_2 f_Y^2 Df \right. \\
 & \left. + 6\alpha_2 (\alpha \gamma_2 + \alpha_1 \delta_2) Df Df_Y \right] + O(h^5) \quad \text{if } \alpha_2 = \beta_2 + \gamma_2 + \delta_2
 \end{aligned}
 \tag{2.14}$$

Obtaining these equations is the most tedious part of the derivation. All that remains is determining values of our arbitrary constants $\{\alpha_i\}$, $\{\beta_i\}$, $\{\gamma_i\}$, $\{\delta_i\}$ and $\{\mu_i\}$ such that

$$\mu_1 k_1 + \mu_2 k_2 + \mu_3 k_3 + \mu_4 k_4$$

corresponds exactly to (2.9) up to and including fourth-order terms in h .

Comparison of (2.11) - (2.14) with (2.9) gives the following relationship between the constants:

$$\mu_1 + \mu_2 + \mu_3 + \mu_4 = 1; \quad (hf) \tag{2.15}$$

$$(\alpha) \mu_2 + (\alpha_1) \mu_3 + (\alpha_2) \mu_4 = 1/2; \quad (h^2 Df) \tag{2.16}$$

$$(\alpha^2) \mu_2 + (\alpha_1^2) \mu_3 + (\alpha_2^2) \mu_4 = 1/3; \quad \left(\frac{h^3}{2} D^2 f \right) \tag{2.17}$$

$$(\alpha \gamma_1) \mu_3 + (\alpha \gamma_2 + \alpha_1 \delta_2) \mu_4 = 1/6; \quad (h^3 f_Y Df) \tag{2.18}$$

$$(\alpha^3) \mu_2 + (\alpha_1^3) \mu_3 + (\alpha_2^3) \mu_4 = 1/4; \quad \left(\frac{h^4}{6} D^2 f \right) \tag{2.19}$$

$$(\alpha^2 \gamma_1) \mu_3 + (\alpha^2 \gamma_2 + \alpha_1^2 \delta_2) \mu_4 = 1/12; \quad \left(\frac{h^4}{2} f_Y D^2 f \right) \tag{2.20}$$

$$(\alpha \gamma_1 \delta_2) \mu_4 = 1/24; \quad (h^4 f_Y^2 Df) \tag{2.21}$$

$$(\alpha \alpha_1 \gamma_1) \mu_3 + \alpha_2 (\alpha \gamma_2 + \alpha_1 \delta_2) \mu_4 = 1/8; \quad (h^4 Df Df_Y) \tag{2.22}$$

In matrix form, these equations are

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \alpha & \alpha_1 & \alpha_2 \\ 0 & \alpha^2 & \alpha_1^2 & \alpha_2^2 \\ 0 & 0 & \alpha\gamma_1 & \alpha\gamma_2 + \alpha_1\delta_2 \\ 0 & \alpha^3 & \alpha_1^3 & \alpha_2^3 \\ 0 & 0 & \alpha^2\gamma_1 & \alpha^2\gamma_2 + \alpha_1^2\delta_2 \\ 0 & 0 & 0 & \alpha\gamma_1\delta_2 \\ 0 & 0 & \alpha\alpha_1\gamma_2 & \alpha_2(\alpha\gamma_2 + \alpha_1\delta_2) \end{bmatrix} \cdot \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/6 \\ 1/4 \\ 1/12 \\ 1/24 \\ 1/8 \end{bmatrix} \quad (2.23)$$

In addition, the following constraints, imposed during the derivation, must hold true -

$$\begin{aligned} \beta &= \alpha \\ \beta_1 + \gamma_1 &= \alpha_1 \\ \beta_2 + \gamma_2 + \delta_2 &= \alpha_2 \end{aligned} \quad (2.24)$$

A method devised by Gill⁽¹⁰⁾ (1951) and set forth in Martin⁽¹¹⁾ is the one most commonly used in engineering applications at present, for solution of first-order differential equations. If

$$\begin{aligned} k_1 &= h \cdot f(x_0, y_0) \\ k_2 &= h \cdot f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \end{aligned} \quad (2.25)$$

$$k_3 = h \cdot f(x_0 + \frac{1}{2}h, y_0 + (-\frac{1}{2} + \frac{1}{\sqrt{2}})k_1 + (1 - \frac{1}{\sqrt{2}})k_2)$$

$$k_4 = h \cdot f(x_0 + h, y_0 + (-\frac{1}{\sqrt{2}})k_2 + (1 + \frac{1}{\sqrt{2}})k_3) ,$$

then

$$y(x_0) + \frac{1}{6} k_1 + \frac{1}{3}(1 - \frac{1}{\sqrt{2}})k_2 + \frac{1}{3}(1 + \frac{1}{\sqrt{2}})k_3 + \frac{1}{6} k_4 \quad (2.26)$$

represents $y(x_0 + h)$ with an error proportional to h^5 , although this error is small. This is Gill's method. As an example of the application of (2.23) and (2.24), the above equations will be analyzed to prove fourth-order accuracy: By inspection of (2.25) and (2.26),

$$\begin{aligned} \alpha &= \frac{1}{2} ; \quad \beta = \frac{1}{2} ; & \mu_1 &= 1/6 \\ \alpha_1 &= \frac{1}{2} ; \quad \beta_1 = -\frac{1}{2} + \frac{1}{\sqrt{2}} ; \quad \gamma_1 = 1 - \frac{1}{\sqrt{2}} ; & \mu_2 &= \frac{1}{3}(1 - \frac{1}{\sqrt{2}}) \\ \alpha_2 &= 1 ; \quad \beta_2 = 0 ; \quad \gamma_2 = -\frac{1}{\sqrt{2}} ; \quad \delta_2 = 1 + \frac{1}{\sqrt{2}} ; & \mu_3 &= \frac{1}{3}(1 + \frac{1}{\sqrt{2}}) \\ & & \mu_4 &= 1/6 \end{aligned}$$

Substitution of these values into (2.23) gives

$$\begin{bmatrix}
 1 & 1 & 1 & 1 \\
 0 & 1/2 & 1/2 & 1 \\
 0 & 1/4 & 1/4 & 1 \\
 0 & 0 & \frac{1}{2}(1 - \frac{1}{\sqrt{2}}) & 1/2 \\
 0 & 1/8 & 1/8 & 1 \\
 0 & 0 & \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) & 1/4 \\
 0 & 0 & 0 & 1/4 \\
 0 & 0 & \frac{1}{4}(1 - \frac{1}{\sqrt{2}}) & 1/2
 \end{bmatrix}
 \begin{bmatrix}
 1/6 \\
 \frac{1}{3}(1 - \frac{1}{\sqrt{2}}) \\
 \frac{1}{3}(1 + \frac{1}{\sqrt{2}}) \\
 1/6
 \end{bmatrix}
 =
 \begin{bmatrix}
 1 \\
 1/2 \\
 1/3 \\
 1/6 \\
 1/4 \\
 1/12 \\
 1/24 \\
 1/8
 \end{bmatrix}$$

Since (2.24) is also satisfied, this verifies that Gill's process is a valid fourth-order process. However, this analysis does nothing to indicate how accurate a fourth-order method it is - no information is given which enables a comparison of the accuracy with that of other procedures. The latter aspect will be treated later.

III. SOLUTION OF THE SWING EQUATION USING

RUNGE-KUTTA FIRST-ORDER DIFFERENTIAL

EQUATION TECHNIQUES

The fastest and most commonly used method of solving second-order differential equations using Runge-Kutta is to break the equation into two simultaneous first-order equations and solve both step by step. As set forth by Hildebrand⁽¹⁶⁾ using the basic procedure of Kutta for $\frac{dy}{dx} = f(x,y)$,

$$\begin{aligned}k_1' &= hf(x_0, y_0) \\k_2' &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1'}{2}\right) \\k_3' &= hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2'}{2}\right) \quad (3.1)\end{aligned}$$

$$k_4' = hf(x_0 + h, y_0 + k_3') \quad \text{and}$$

$$y_1 = y(x_0 + h) = y_0 + \frac{1}{6}(k_1' + 2k_2' + 2k_3' + k_4')$$

the method becomes as follows when applied to the equations

$$\frac{dy}{dx} = F(x, y, u) \quad \text{and} \quad (3.2)$$

$$\frac{du}{dx} = G(x, y, u) : \quad (3.3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad \text{and} \quad (3.4)$$

$$u_{n+1} = u_n + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4); \text{ where} \quad (3.5)$$

$$\begin{aligned} k_1 &= hF(x_n, y_n, u_n) \\ k_2 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, u_n + \frac{m_1}{2}\right) \\ k_3 &= hF\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, u_n + \frac{m_2}{2}\right) \\ k_4 &= hF(x_n + h, y_n + k_3, u_n + m_3); \end{aligned} \quad (3.6)$$

and

$$\begin{aligned} m_1 &= hG(x_n, y_n, u_n) \\ m_2 &= hG\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}, u_n + \frac{m_1}{2}\right) \\ m_3 &= hG\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}, u_n + \frac{m_2}{2}\right) \\ m_4 &= hG(x_n + h, y_n + k_3, u_n + m_3). \end{aligned} \quad (3.7)$$

For an equation of the same form as the basic swing equation, say

$$\frac{d^2 y}{dx^2} = f(x, y, y'), \quad (3.8)$$

the simultaneous equations are

$$\frac{dy}{dx} = u = g(x, y, u) \quad \text{and} \quad \frac{du}{dx} = f(x, y, u). \quad (3.9)$$

In order to compare different Runge-Kutta methods applied here, in the same manner as in (3.2) to (3.7), it is expedient to

derive a conversion technique. Suppose we are given a first-order differential equation method such as

$$k_1' = hf(x_0, y_0)$$

$$k_2' = hf(x_0 + \alpha_1 h, y_0 + \frac{a_{11}}{2} k_1')$$

$$k_3' = hf(x_0 + \alpha_2 h, y_0 + \frac{a_{21}}{2} k_1' + \frac{b_{21}}{2} k_2')$$

$$k_4' = hf(x_0 + \alpha_3 h, y_0 + \frac{a_{31}}{2} k_1' + \frac{b_{31}}{2} k_2' + \frac{c_{31}}{2} k_3'),$$

and

$$y_1 = y_0 + (\frac{\mu_1}{2} k_1' + \frac{\mu_2}{2} k_2' + \frac{\mu_3}{2} k_3' + \frac{\mu_4}{2} k_4'),$$

where

$$y_0 = y(x_0), \quad y_1 = y(x_0 + h), \quad \text{and} \quad \frac{dy}{dx} = f(x, y). \quad (3.10)$$

Expressed in tabular form, this is:

x_i	y_i	$k_i' = hf(x_i, y_i)$
x_0	y_0	k_1'
$x_0 + \alpha_1 h$	$y_0 + \frac{a_{11}}{2} k_1'$	k_2'
$x_0 + \alpha_2 h$	$y_0 + \frac{a_{21}}{2} k_1' + \frac{b_{21}}{2} k_2'$	k_3'
$x_0 + \alpha_3 h$	$y_0 + \frac{a_{31}}{2} k_1' + \frac{b_{31}}{2} k_2' + \frac{c_{31}}{2} k_3'$	k_4'
$x_1 = x_0 + h$	$y_1 = y_0 + k$	$k = \frac{\mu_1}{2} k_1' + \frac{\mu_2}{2} k_2' + \frac{\mu_3}{2} k_3' + \frac{\mu_4}{2} k_4'$

(3.11)

Using the generalized method of (3.11), it is now possible to apply the same procedure to (3.9) as was done in (3.2) to (3.7).

Let

$$\begin{aligned} m_1 &= hf(x_0, y_0, u_0) \\ m_2 &= hf(x_0 + \alpha_1 h, y_0 + \frac{a_{11}}{2} n_1, u_0 + \frac{a_{11}}{2} m_1) \\ m_3 &= hf(x_0 + \alpha_3 h, y_0 + \frac{a_{21}}{2} n_1 + \frac{b_{21}}{2} n_2, \\ &\quad u_0 + \frac{a_{21}}{2} m_1 + \frac{b_{21}}{2} m_2) \end{aligned} \tag{3.12}$$

$$\begin{aligned} m_4 &= hf(x_0 + \alpha_3 h, y_0 + \frac{a_{31}}{2} n_1 + \frac{b_{31}}{2} n_2 + \frac{c_{31}}{2} n_3, \\ &\quad u_0 + \frac{a_{31}}{2} m_1 + \frac{b_{31}}{2} m_2 + \frac{c_{31}}{2} m_3), \end{aligned}$$

and

$$\begin{aligned} n_1 &= hg(x_0, y_0, u_0) \\ n_2 &= hg(x_0 + \alpha_1 h, y_0 + \frac{a_{11}}{2} n_1, u_0 + \frac{a_{11}}{2} m_1) \\ n_3 &= hg(x_0 + \alpha_2 h, y_0 + \frac{a_{21}}{2} n_1 + \frac{b_{21}}{2} n_2, \\ &\quad u_0 + \frac{a_{21}}{2} m_1 + \frac{b_{21}}{2} m_2) \\ n_4 &= hg(x_0 + \alpha_3 h, y_0 + \frac{a_{31}}{2} n_1 + \frac{b_{31}}{2} n_2 + \frac{c_{31}}{2} n_3, \\ &\quad u_0 + \frac{a_{31}}{2} m_1 + \frac{b_{31}}{2} m_2 + \frac{c_{31}}{2} m_3), \end{aligned} \tag{3.13}$$

so that

$$y_1 = y_0 + \frac{\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3 + \mu_4 n_4}{2} \quad (3.14)$$

and

$$u_1 = u_0 + \frac{\mu_1 n_1 + \mu_2 n_2 + \mu_3 n_3 + \mu_4 n_4}{2} . \quad (3.15)$$

Since $g(x,y,u) = u$ in this case, (3.9), it follows that

$$\begin{aligned} n_1 &= hu_0 \\ n_2 &= hu_0 + h \frac{a_{11}}{2} m_1 \\ n_3 &= hu_0 + h \frac{a_{21}}{2} m_1 + h \frac{b_{21}}{2} m_2 \\ n_4 &= hu_0 + h \frac{a_{31}}{2} m_1 + h \frac{b_{31}}{2} m_2 + h \frac{c_{31}}{2} m_3 , \end{aligned} \quad (3.16)$$

so that (3.12) becomes:

$$\begin{aligned} m_1 &= hf(x_0, y_0, u_0) \\ m_2 &= hf(x_0 + \alpha_1 h, y_0 + \frac{a_{11}}{2} hu_0, u_0 + \frac{a_{11}}{2} m_1) \\ m_3 &= hf(x_0 + \alpha_2 h, y_0 + (\frac{a_{21} + b_{21}}{2}) hu_0 \\ &\quad + \frac{b_{21}}{2} h \frac{a_{11}}{2} m_1, u_0 + \frac{a_{21}}{2} m_1 + \frac{b_{21}}{2} m_2) \\ m_4 &= hf(x_0 + \alpha_3 h, y_0 + \frac{a_{31} + b_{31} + c_{31}}{2} hu_0 + (\frac{b_{31}}{2} h \frac{a_{11}}{2} \\ &\quad + \frac{c_{31}}{2} h \frac{a_{21}}{2}) m_1 + \frac{c_{31}}{2} h \frac{b_{21}}{2} m_2, \\ &\quad u_0 + \frac{a_{31}}{2} m_1 + \frac{b_{31}}{2} m_2 + \frac{c_{31}}{2} m_3) , \end{aligned} \quad (3.17)$$

and (3.14) becomes:

$$y_1 = y_0 + \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{2} hu_0 + \frac{h}{2.2} \left[(\mu_2 a_{11} + \mu_3 a_{21} + \mu_4 a_{31})m_1 + (\mu_3 b_{21} + \mu_4 b_{31})m_2 + \mu_4 c_{31}m_3 \right] \quad (3.18)$$

If we now define $\{k_i\} = \frac{h}{2} \{m_i\}$ and set $v_i = hu_i$, the procedure reduces to:

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_0, y_0, u_0) \\ k_2 &= \frac{h^2}{2} f(x_0 + \alpha_1 h, y_0 + \frac{a_{11}}{2} v_0, u_0 + \frac{a_{11} k_1}{h}) \\ k_3 &= \frac{h^2}{2} f(x_0 + \alpha_2 h, y_0 + \frac{a_{21} + b_{21}}{2} v_0 + \frac{a_{21} b_{21}}{2} k_1, \\ &\quad u_0 + a_{21} \frac{k_1}{h} + \frac{b_{21} k_2}{h}) \\ k_4 &= \frac{h^2}{2} f(x_0 + \alpha_3 h, y_0 + \frac{a_{31} + b_{31} + c_{31}}{2} v_0 \\ &\quad + \frac{a_{11} b_{31} + a_{21} c_{31}}{2} k_1 + \frac{b_{21} c_{31}}{2} k_2, \\ &\quad u_0 + \frac{a_{31}}{h} k_1 + \frac{b_{31}}{h} k_2 + \frac{c_{31}}{h} k_3), \end{aligned}$$

with

$$y_1 = y_0 + \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{2} v_0 + \left[(\mu_2 a_{11} + \mu_3 a_{21} + \mu_4 a_{31}) k_1 + (\mu_3 b_{21} + \mu_4 b_{31}) k_2 + (\mu_4 c_{31}) k_3 \right] / 2 ,$$

and

$$v_1 = v_0 + \mu_1 k_1 + \mu_2 k_2 + \mu_3 k_3 + \mu_4 k_4 .$$

In tabular form, the procedure is as follows:

x_i	y_i	$v_i = hy_i'$	k_i
x_0	y_0	v_0	k_1
$x_0 + \alpha_1 h$	$y_0 + \frac{a_{11}}{2} v_0$	$v_0 + a_{11} k_1$	k_2
$x_0 + \alpha_2 h$	$y_0 + \frac{a_{21} + b_{21}}{2} v_0 + \frac{a_{11} b_{21}}{2} k_1$	$v_0 + a_{21} k_1 + b_{21} k_2$	k_3
$x_0 + \alpha_3 h$	$y_0 + \frac{a_{31} + b_{31} + c_{31}}{2} v_0 + \frac{a_{11} b_{31} + a_{21} c_{31}}{2} k_1 + \frac{b_{21} c_{31}}{2} k_2$	$v_0 + a_{31} k_1 + b_{31} k_2 + c_{31} k_3$	k_4
$x_1 = x_0 + h$	$y_1 = y_0 + \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4}{2} v_0 + \frac{k}{2}$ $k = (\mu_2 a_{11} + \mu_3 a_{21} + \mu_4 a_{31}) k_1 + (\mu_3 b_{21} + \mu_4 b_{31}) k_2 + (\mu_4 c_{31}) k_3$	$v_1 = v_0 + k'$ $k' = \mu_1 k_1 + \mu_2 k_2 + \mu_3 k_3 + \mu_4 k_4$	

(3.19)

In each case $k_i = \frac{h^2}{2} f(x_i, y_i, y_i') .$

Given any Runge-Kutta procedure to solve a first-order differential equation

$$\frac{dy}{dx} = f(x,y); \quad y(x_0) = y_0 ,$$

by comparing coefficients with those of the general procedure set forth in (3.11), it is possible to use (3.19) to obtain an analogous method for use in solving

$$\frac{d^2y}{dx^2} = f(x,y,y'); \text{ with } y(x_0) = y_0 \text{ and } y'(x_0) = y'_0 .$$

Note that, in contrast to Hildebrand's method (3.4) - (3.7), only four k's need be evaluated. This amounts to a considerable saving in computation time.

3.1 Conversion of Four Basic Runge-Kutta Processes

The most commonly used first-order differential equation methods are outlined by Martin. These are:

- a) Kutta's Simpson's rule;
- b) Gill's process;
- c) Strachey's process;
- d) Boulton's process.

Kutta's Simpson's rule has been adapted for second-order differential equations by Hildebrand⁽¹⁶⁾. For the equation

$$\frac{d^2y}{dx^2} = f(x,y,y'), \quad y(x_0) = y_0, y'(x_0) = y'_0, \quad hy_i = v_i \quad (3.20)$$

if

$$\begin{aligned}
 k_1 &= \frac{h^2}{2} f(x_0, y_0, \frac{v_0}{h}), \\
 k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0, \frac{v_0 + k_1}{h}), \\
 k_3 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + y_0 + \frac{1}{2}v_0 + \frac{1}{2}k_1, \frac{v_0 + k_2}{h}), \text{ and} \\
 k_4 &= \frac{h^2}{2} f(x_0 + h, y_0 + v_0 + k_2, \frac{v_0 + 2k_3}{h}),
 \end{aligned} \tag{3.21}$$

then

$$\begin{aligned}
 y_1 &= y_0 + v_0 + \frac{1}{3} (k_1 + k_2 + k_3), \text{ and} \\
 v_1 &= v_0 + \frac{1}{3} (k_1 + 2k_2 + 2k_3 + k_4).
 \end{aligned}$$

Gill's process is outlined in its first-order form in (2.25) and (2.26). By comparison with Table (3.11),

$$\begin{aligned}
 \mu_1 &= \frac{1}{3}; \\
 \mu_2 &= \frac{2}{3} (1 - \frac{1}{\sqrt{2}}); \quad \alpha_1 = \frac{1}{2}; \quad \frac{a_{11}}{2} = \frac{1}{2} \\
 \mu_3 &= \frac{2}{3} (1 + \frac{1}{\sqrt{2}}); \quad \alpha_2 = \frac{1}{2}; \quad \frac{a_{21}}{2} = -\frac{1}{2} + \frac{1}{\sqrt{2}}; \quad \frac{b_{21}}{2} = 1 - \frac{1}{\sqrt{2}} \\
 \mu_4 &= \frac{1}{3}; \quad \alpha_3 = 1; \quad \frac{a_{31}}{2} = 0; \quad \frac{b_{31}}{2} = -\frac{1}{\sqrt{2}}; \quad \frac{c_{31}}{2} = 1 + \frac{1}{\sqrt{2}}.
 \end{aligned} \tag{3.22}$$

These constants and (3.11) completely characterize the process.

Substitution of (3.22) into (3.11) gives the required result:

$$\begin{aligned}
 k_1 &= \frac{h^2}{2} f(x_0, y_0, \frac{v_0}{h}), \\
 k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0, \frac{v_0 + k_1}{h}), \\
 k_3 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0 + (1 - \frac{1}{\sqrt{2}})k_1, \\
 &\quad \frac{v_0 + (-1 + \sqrt{2})k_1 + (2 - \sqrt{2})k_2}{h}), \\
 k_4 &= \frac{h^2}{2} f(x_0 + h, y_0 + v_0 + k_2, \frac{v_0 - \sqrt{2}k_2 + (2 + \sqrt{2})k_3}{h}),
 \end{aligned}$$

and

$$\begin{aligned}
 y_1 &= y_0 + v_0 + \frac{1}{3} k_1 + (1 - \frac{1}{\sqrt{2}})k_2 + (1 + \frac{1}{\sqrt{2}})k_3 ; \\
 v_1 &= v_0 + \frac{1}{3} [k_1 + 2(1 - \frac{1}{\sqrt{2}})k_2 + 2(1 + \frac{1}{\sqrt{2}})k_3 + k_4] . \quad (3.23)
 \end{aligned}$$

Strachey's process

If $y' = f(x, y)$, and

$$\begin{aligned}
 k_1' &= hf(x_0, y_0) \\
 k_2' &= hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1') \\
 k_3' &= hf(x_0 + \frac{1}{2}h, y_0 - \frac{1}{2}k_1' + k_2') \\
 k_4' &= hf(x_0 + h, y_0 + \frac{1}{2}k_2 + \frac{1}{2}k_3'),
 \end{aligned} \quad (3.24)$$

then

$$y_1 = y_0 + \frac{1}{6} (k_1' + 3k_2' + k_3' + k_4') + O(h^5).$$

In a similar manner as before, substitution of the describing constants into (3.11) gives the second-order procedure:

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_0, y_0, \frac{v_0}{h}), \\ k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0, \frac{v_0 + k_1}{h}), \\ k_3 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0 + k_1 \frac{v_0 - k_1 + 2k_2}{h}), \\ k_4 &= \frac{h^2}{2} f(x_0 + h, y_0 + v_0 + k_2, \frac{v_0 + k_2 + k_3}{h}), \end{aligned} \quad (3.25)$$

with

$$\begin{aligned} y_1 &= y_0 + v_0 + \frac{1}{6} (2k_1 + 3k_2 + k_3); \text{ and} \\ v_1 &= v_0 + \frac{1}{3} (k_1 + 3k_2 + k_3 + k_4). \end{aligned}$$

Boulton's process

For a first order differential equation Boulton's process is characterized by

$$\begin{aligned} k_1' &= hf(x_0, y_0) \\ k_2' &= hf(x_0 + \frac{1}{3}h, y_0 + \frac{1}{3}k_1') \end{aligned}$$

$$k_3' = hf(x_0 + \frac{2}{3}h, y_0 - \frac{1}{3}k_1' + k_2') \quad (3.26)$$

$$k_4' = hf(x_0 + h, y_0 + k_1' - k_2' + k_3')$$

and

$$y_1 = y_0 + \frac{1}{8}(k_1' + 3k_2' + 3k_3' + k_4').$$

The corresponding second order procedure is:

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_0, y_0, \frac{v_0}{h}), \\ k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{3}h, y_0 + \frac{1}{3}v_0, \frac{v_0 + \frac{2}{3}k_1}{h}) \quad (3.27) \\ k_3 &= \frac{h^2}{2} f(x_0 + \frac{2}{3}h, y_0 + \frac{2}{3}v_0 + \frac{2}{3}k_1, \frac{v_0 - \frac{2}{3}k_1 + 2k_2}{h}), \\ k_4 &= \frac{h^2}{2} f(x_0 + h, y_0 + v_0 - \frac{4}{3}k_1 + 2k_2, \\ &\quad \frac{v_0 + 2k_1 - 2k_2 + 2k_3}{3}), \end{aligned}$$

with

$$y_1 = y_0 + v_0 + \frac{1}{4}(k_1 + 2k_2 + k_3); \quad \text{and}$$

$$v_1 = v_0 + \frac{1}{4}(k_1 + 3k_2 + 3k_3 + k_4).$$

3.2 Solution of a Sample Problem

A comparison of the suitability of these methods to the swing equation will be done using the most common example, one from Kimbark⁽¹⁾.

Example 3.1

A 25 MVA, 60 c/s waterwheel generator delivers 20 MW over a double-circuit transmission line to a large metropolitan system, which may be regarded as an infinite bus (Fig. 3.1). The inertia constant H is 2.76 MJ/MVA and the direct-axis transient reactance of the generator is 0.30 p.u. The transmission circuits have negligible resistance, and each has a reactance of 0.2 p.u. on a 25 MVA base. The voltage behind the transient reactance of the generator is 1.03 p.u. and the voltage of the metropolitan system is 1.00 p.u. A 3-phase short circuit occurs at the middle of one transmission circuit. Obtain swing curves for the generator assuming a clearing time of 0.60 second.

The amplitudes of the power-angle curves obtained after network reduction are:

<u>Condition</u>	<u>Amplitude</u>
Pre-fault	2.58 p.u.
During fault	0.936 p.u.
Post-fault	2.06 p.u.

Two equations result from this data, after a change in variable:

a) valid during fault:

$$\frac{d^2 \delta}{dT^2} = 0.855 - 1.0 \sin \delta \quad (3.28)$$

b) valid after clearing:

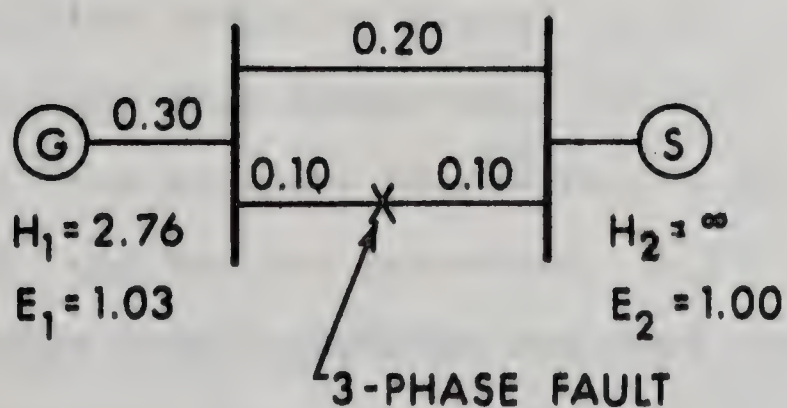
$$\frac{d^2 \delta}{dT^2} = 0.855 - 2.201 \sin \delta . \quad (3.29)$$

In each case,

δ is the displacement angle of the machine rotor with respect to an imaginary reference axis rotating at synchronous speed;

FIG. 3.1

SINGLE-LINE DIAGRAM OF A POWER SYSTEM SHOWING REACTANCES IN PER-UNIT.



T is proportional to elapsed time measured from the beginning of the disturbance.

From the pre-fault data, it is found that at $T = 0$,

$$\delta_o = 0.316 \text{ electrical radians}$$

$$\left. \frac{d\delta}{dt} \right|_o = \omega_o = 0. \text{ el. radians/second} \quad (3.30)$$

Since a damping term is not included, both equations are of the general form

$$\frac{d^2\delta}{dT^2} = f(\delta); \text{ with } \delta(0) = \delta_o; \quad \delta'(0) = \omega_o.$$

Here, $f(\delta) = 0.855 - C \sin \delta$, where C has one value during the fault and another after the fault is cleared.

The problem is quickly and accurately solved on a digital computer, using the flow chart shown in Fig. 3.2. The four numerical integration methods compared are:

- a) the Kutta procedure, (3.21),
- b) the Gill procedure, (3.23),
- c) the Strachey procedure, (3.25), and
- d) the Boulton procedure, (3.27).

Seven time increments are used in each case, varying from $h = 0.20$ sec. to $h = .003125$ sec., to give an idea of the relative stability of each method. For each value of h integration is carried out from $t = 0$ to $t = 1.0$ second. The

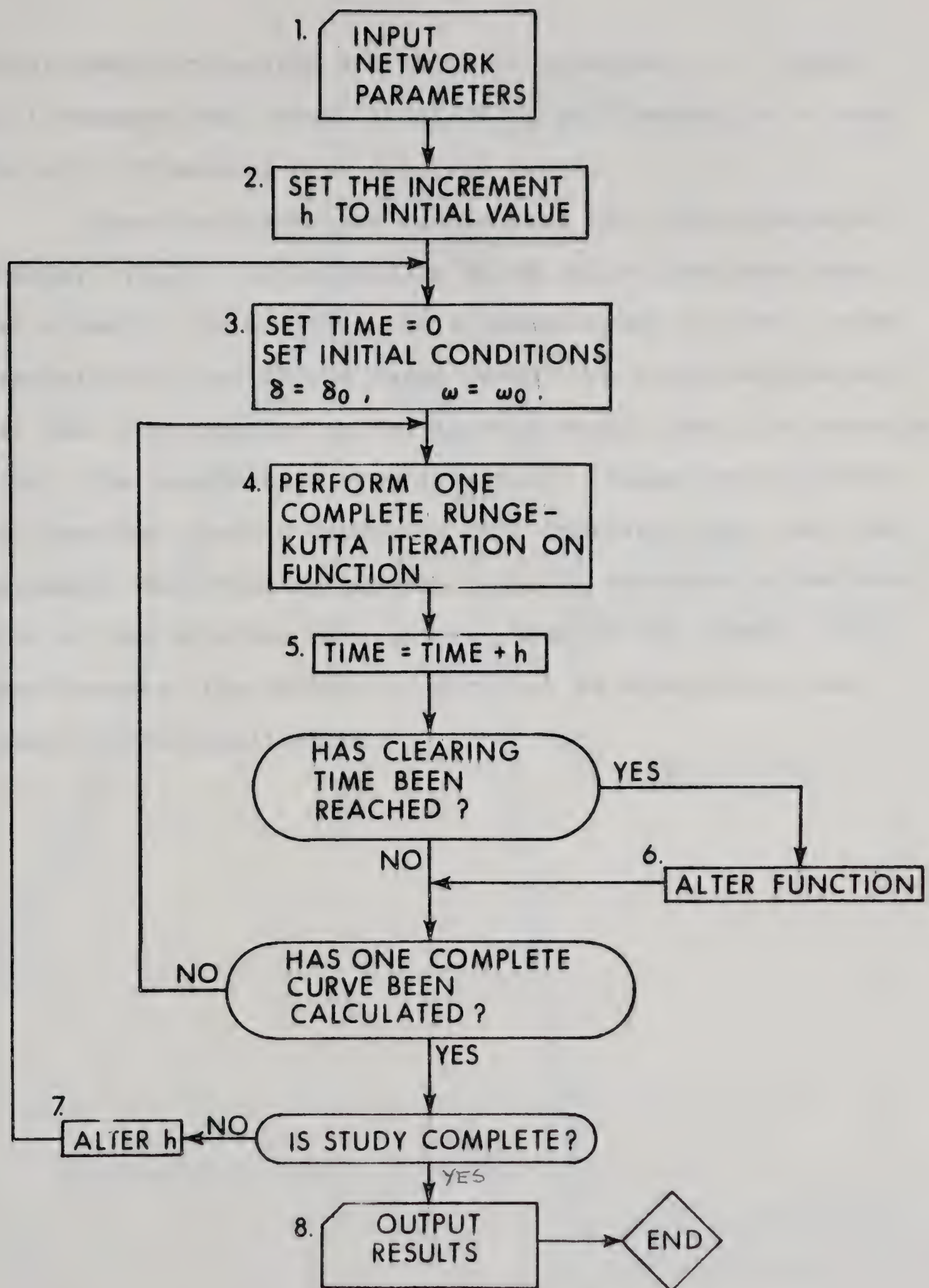


FIG. 3.2 ACCURACY COMPARISON FLOW CHART OF EXAMPLE 3.1

actual computer results are shown in Appendix (C). Figure (3.3) compares the curves obtained by each method for a step size of 0.20 seconds with the true curve.

From the figure, it can be seen that the process of Strachey, (3.25), gives results which follow the true curve most closely. In addition, this method tends to give a more "pessimistic" view of the swing curve; the other methods suggest that the equation solved is more stable than, in actuality, it is. The pessimistic view is safer. Inspection of the direct computer results (Appendix (C)) indicates that, as h is decreased, the Strachey process tends to converge to the true value of the solution more quickly than do the others. For these reasons, the process of Strachey is superior to the others in this application.

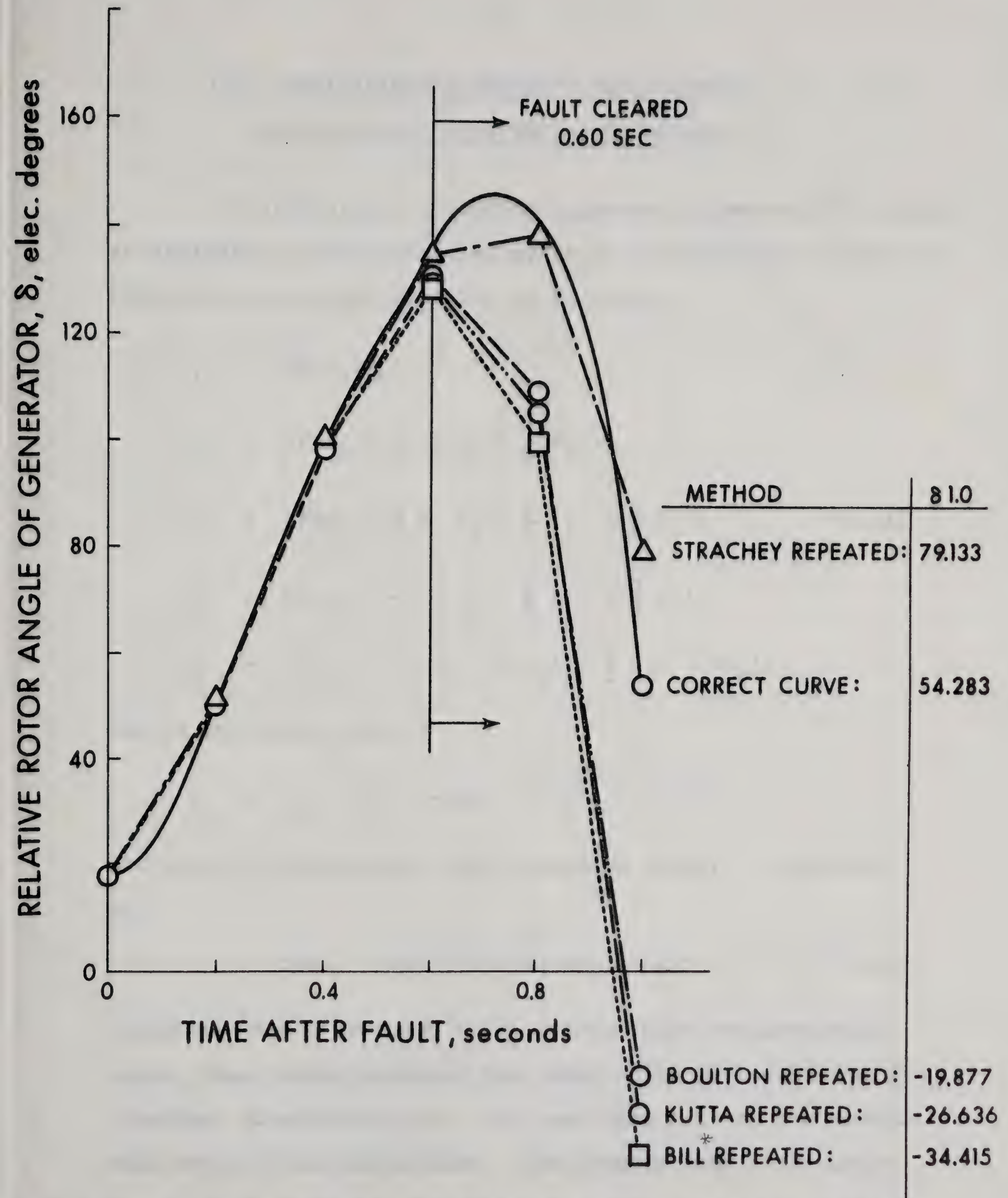


FIG. 3.3 SWING CURVES OF EXAMPLE 3.1 - CLEARING TIME 0.60 SEC

3.3 Application of Merson's Variation to Second-Order Differential Equations

An alternative procedure suggested by Merson⁽¹²⁾ allows an estimate of the truncation error to be computed at each step. This is, to evaluate five k's as follows:

$$\begin{aligned}k_1' &= hf(x_0, y_0) \\k_2' &= hf(x_0 + \frac{1}{3}h, y_0 + \frac{1}{6}k_1') \\k_3' &= hf(x_0 + \frac{1}{3}h, y_0 + \frac{1}{6}k_1' + \frac{1}{6}k_2') \\k_4' &= hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{8}k_1' + \frac{3}{8}k_3') \\k_5' &= hf(x_0 + h, y_0 + \frac{1}{2}k_1' - \frac{3}{2}k_3' + 2k_4'),\end{aligned}\tag{3.31}$$

and to integrate using

$$y_1 = y_0 + \frac{1}{6}(k_1' + 4k_4' + k_5') + O(h^5),$$

for which an estimate of the truncation error, ϵ , is given by

$$30\epsilon = 2k_1' - 9k_3' + 8k_4' - k_5'\tag{3.32}$$

If at any step ϵ is found to be greater than the acceptable error, then Merson suggests the integration be repeated with h halved; alternatively if ϵ is less than $1/32$ of the acceptable error, h may be doubled. The process, set forth above

for a first-order differential equation, can lead to an overall gain of efficiency⁽¹²⁾, despite the additional evaluation of f required at each step.

In order to adapt this process so that it is suitable for second-order equations, the procedure of (3.11) - (3.19) must be extended to encompass expressions for ϵ and k_5 .

$$\text{At } x = x_0 + \alpha_4 h, \text{ let } y = y_0 + \frac{a_{41}}{2} k_1' + \frac{b_{41}}{2} k_2' + \frac{c_{41}}{2} k_3' + \frac{d_{41}}{2} k_4'$$

and let

$$k = \frac{\mu_1}{2} k_1' + \frac{\mu_2}{2} k_2' + \frac{\mu_3}{2} k_3' + \frac{\mu_4}{2} k_4' + \frac{\mu_5}{2} k_5'.$$

To (3.12) and (3.16) we must add

$$\begin{aligned} m_5 = & hf(x_0 + \alpha_4 h, y_0 + \frac{a_{41}}{2} n_1 + \frac{b_{41}}{2} n_2 + \frac{c_{41}}{2} n_3 \\ & + \frac{d_{41}}{2} n_4, u_0 + \frac{a_{41}}{2} m_1 + \frac{b_{41}}{2} m_2 + \frac{c_{41}}{2} m_3 \\ & + \frac{d_{41}}{2} m_4) \end{aligned}$$

and

$$n_5 = hu_0 + \frac{ha_{41}}{2} m_1 + \frac{hb_{41}}{2} m_2 + \frac{hc_{41}}{2} m_3 + \frac{hd_{41}}{2} m_4.$$

In place of (3.18) we now have

$$\begin{aligned} y_1 = & y_0 + \frac{\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5}{2} hu_0 + \frac{h}{2} \left(\frac{\mu_2 a_{11} + \mu_3 a_{21} + \mu_4 a_{31} + \mu_5 a_{41}}{2} m_1 \right. \\ & \left. + \frac{\mu_3 b_{21} + \mu_4 b_{31} + \mu_5 b_{41}}{2} m_2 + \frac{\mu_4 c_{31} + \mu_5 c_{41}}{2} m_3 + \frac{\mu_5 d_{41}}{2} m_4 \right) \end{aligned}$$

so that, if

$$k_5 = \frac{h^2}{2} f(x_0 + \alpha_4 h, y_0 + \frac{a_{41} + b_{41} + c_{41} + d_{41}}{2} v_0 \\ + \frac{a_{11}b_{41} + a_{21}c_{41} + a_{31}d_{41}}{2} k_1 + \frac{b_{21}c_{41} + b_{31}d_{41}}{2} k_2 \\ + \frac{c_{31}d_{41}}{2} k_3, u_0 + \frac{a_{41}}{2} k_1 + \frac{b_{41}}{2} k_2 + \frac{c_{41}}{2} k_3 + \frac{d_{41}}{2} k_4)$$

then

$$y_1 = y_0 + \sum_{i=1}^5 \frac{\mu_i}{2} v_0 + \frac{\mu_2 a_{11} + \mu_3 a_{21} + \mu_4 a_{31} + \mu_5 a_{41}}{2} k_{41} \\ + \frac{\mu_3 b_{21} + \mu_4 b_{31} + \mu_4 b_{41}}{2} k_2 + \frac{\mu_4 c_{31} + \mu_5 c_{41}}{2} k_3 \\ + \frac{\mu_5 d_{41}}{2} k_4 ,$$

and

$$v_1 = v_0 + \sum_{i=1}^5 \mu_i k_i .$$

Also from (3.32), using $k_i = \frac{h}{2} m_i$,

$$30\epsilon(u_1) = 2m_1 - 9m_3 + 8m_4 - m_5 , \quad \text{and}$$

$$h \cdot \epsilon(u_1) = \frac{2k_1 - 9k_3 + 9k_4 - k_5}{15} .$$

Using (3.32), (3.16) and the fact that $k_i = \frac{h}{2} m_i$ the expression for error in y_1 may also be found:

$$\begin{aligned}
 30\epsilon(y_1) &= 2n_1 - 9n_3 + 8n_4 - n_5 \\
 &= \frac{h}{2} (-9a_{21}m_1 - 9b_{21}m_2 + 8a_{31}m_1 + 8b_{31}m_2 \\
 &\quad + 8c_{31}m_3 - a_{41}m_2 - c_{41}m_3 - d_{41}m_4)
 \end{aligned}$$

and

$$\epsilon(y_1) = \frac{(-9a_{21}+8a_{31}-a_{41})k_1 + (-9b_{21}+8b_{31}-b_{41})k_2 + (8c_{31}-c_{41})k_3 + (-d_{41})k_4}{30}$$

The values of the $\{a_i\}$, $\{b_i\}$, $\{c_i\}$, $\{d_i\}$, $\{\mu_i\}$, from equation (3.31), are as follows:

$$\begin{aligned}
 \mu_1 &= \frac{1}{3}; \frac{a_{11}}{2} = \frac{1}{3}; & \alpha_1 &= \frac{1}{3}; \\
 \mu_2 &= 0; \frac{a_{21}}{2} = \frac{1}{6}; \frac{b_{21}}{2} = \frac{1}{6}; & \alpha_2 &= \frac{1}{3}; \\
 \mu_3 &= 0; \frac{a_{31}}{2} = \frac{1}{8}; \frac{b_{31}}{2} = 0; \frac{c_{31}}{2} = \frac{3}{8}; & \alpha_3 &= \frac{1}{2}; \quad (3.33) \\
 \mu_4 &= \frac{4}{3}; \frac{a_{41}}{2} = \frac{1}{2}; \frac{b_{41}}{2} = 0; \frac{c_{41}}{2} = -\frac{3}{2}; \frac{d_{41}}{2} = 2; & \alpha_4 &= 1. \\
 \mu_5 &= \frac{1}{3}.
 \end{aligned}$$

Hence, the procedure to be applied to $y'' = f(x, y, y')$, for one iteration, is:

$$\begin{aligned}
 k_1 &= \frac{h^2}{2} f(x_0, y_0, \frac{v_0}{h}) \\
 k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{3}h, y_0 + \frac{1}{3}v_0, \frac{v_0 + \frac{2}{3}k_1}{h})
 \end{aligned}$$

$$k_3 = \frac{h^2}{2} f(x_0 + \frac{1}{3}h, y_0 + \frac{1}{3}v_0 + \frac{1}{9}k_1, \frac{v_0 + \frac{1}{3}k_1 + \frac{1}{3}k_2}{h}) \quad (3.34)$$

$$k_4 = \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}v_0 + \frac{1}{8}k_1 + \frac{1}{8}k_2, \frac{v_0 + \frac{1}{4}k_1 + \frac{3}{4}k_3}{h})$$

$$k_5 = \frac{h^2}{2} f(x_0 + h, y_0 + v_0 - \frac{1}{2}k_2 + \frac{3}{2}k_3, \frac{v_0 + k_1 - 3k_3 + 4k_4}{h})$$

and

$$y_1 = y_0 + v_0 + \frac{1}{3}(k_1 + 2k_4),$$

$$v_i = hu_i ;$$

$$u_1 = u_0 + \frac{1}{3}(\frac{k_1 + 4k_4 + k_5}{h}); \text{ with}$$

$$\epsilon(y_1) \approx \frac{-2k_1 - 3k_2 + 9k_3 - 4k_4}{30}; \text{ and}$$

$$\epsilon(u_1) \approx \frac{2k_1 - 9k_3 + 8k_4 - k_5}{15h}.$$

Example 3.2

As a means of comparing the accuracy of the "Merson Repeated" procedure with methods described previously, repeat Example 3.1, assuming a fault clearing time of 0.6 seconds, using the iteration method of (3.34).

Again, the equation to be solved is of the form

$$\frac{d^2\delta}{dT^2} = f(\delta), \quad \delta(0) = \delta_0, \quad \frac{d\delta}{dt}(0) = \omega_0 = \frac{v_0}{h} \quad (3.35)$$

Hence, (3.34) simplifies to

$$\begin{aligned}
 k_1 &= \frac{h^2}{2} f(\delta_0) \\
 k_2 &= \frac{h^2}{2} f\left(\delta_0 + \frac{1}{3} v_0\right) \\
 k_3 &= \frac{h^2}{2} f\left(\delta_0 + \frac{1}{3} v_0 + \frac{k_1}{9}\right) \\
 k_4 &= \frac{h^2}{2} f\left(\delta_0 + \frac{1}{2} v_0 + \frac{k_1}{8} + \frac{k_2}{8}\right) \\
 k_5 &= \frac{h^2}{2} f\left(\delta_0 + v_0 - \frac{k_2}{2} + \frac{3k_3}{2}\right)
 \end{aligned}$$

so that at the end of the first iteration, (3.36)

$$\begin{aligned}
 \delta_1 &= \delta_0 + v_0 + \frac{1}{3} (k_1 + 2k_4) \\
 v_1 &= v_0 + \frac{1}{3} (k_1 + 4k_4 + k_5), \quad (\omega_1 = \frac{v_1}{h})
 \end{aligned}$$

and the truncation errors are given by

$$\begin{aligned}
 \epsilon(\delta_1) &\approx \frac{-2k_1 - 3k_2 + 9k_3 - 4k_4}{30} ; \\
 \epsilon(v_1) &\approx \frac{2k_1 - 9k_3 + 8k_4 - k_5}{15} .
 \end{aligned}$$

The flow chart used to program the problem for a digital computer is the same as the one of Fig. 3.2. Step 4 in that figure encompasses the above operations (3.36).

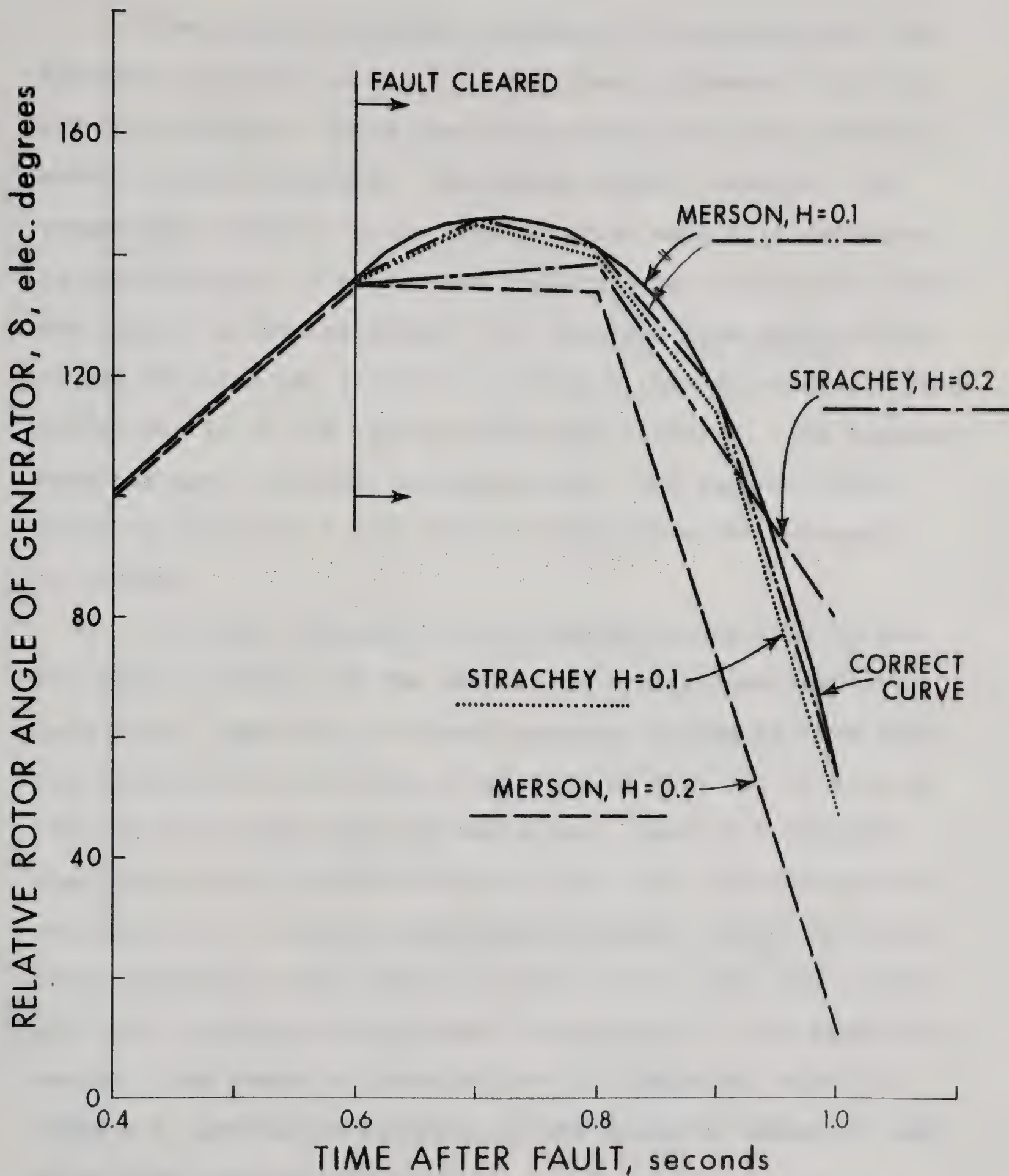


FIG. 3.4 SWING CURVES OF EXAMPLE 3.2 - CLEAR TIME 0.60 SEC.

The Merson iteration procedure is compared with the Strachey procedure in Fig. 3.4 for time increments h of 0.10 and 0.20 seconds. Using the larger value of h , the Strachey method is more accurate. The Merson method, however, converges more rapidly to the true solution when h is decreased. It must be borne in mind, when comparing the procedures, that the latter is less efficient - it requires five evaluations of the function per iteration, one more than Strachey's. This proves costly in the case of the swing equation. The computer requires many addition, multiplication, and exponentiation steps to calculate a sine function each time the subroutine is entered.

The real advantage of the Merson method lies in the available estimates of the truncation errors committed at each step. Appendix (C) gives the error values at each step for the example, and some are plotted in Fig. 3.5 to give an idea of their magnitude and variation. Table 3.1 compares the results of a computer program using the interval changing criterion of (3.32a), as set forth by Merson (with the second order integration set forth in (3.34) and (3.36)), and the results of the basic program used in Example 3.1, for Strachey's method. The number of entries into the function, given in Table 3.1, provide an estimate of the relative amount of computer time required for each.

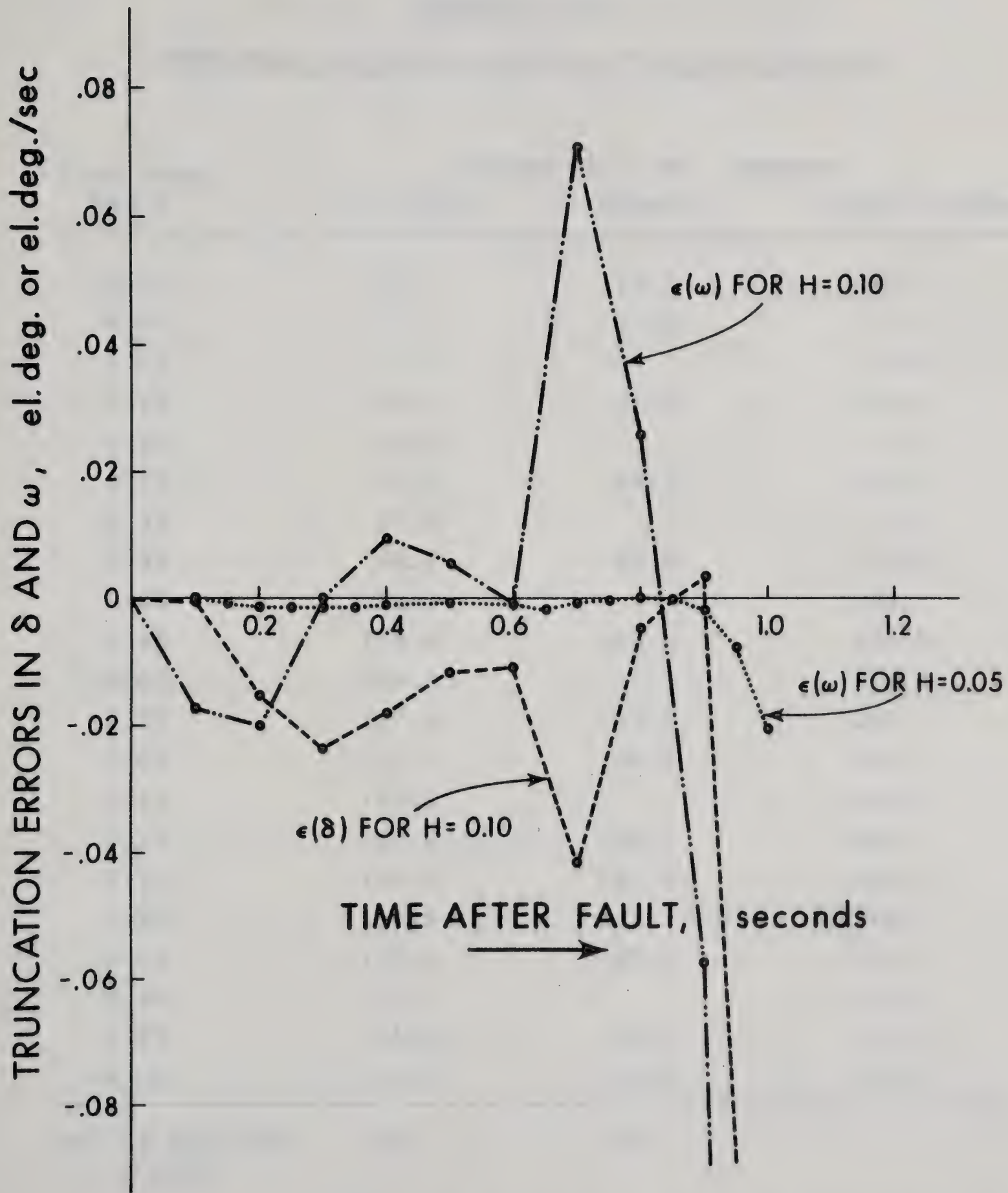


FIG. 3.5 CALCULATED TRUNCATION ERROR VALUES -
EXAMPLE 3.2

Table 3.1

Example 3.2

Some Data for Fault Clearing Time 0.60 Seconds

Time after Fault	Values of δ , el. degrees		
	Strachey	Merson	Actual Values
0.00	18.1	18.1	18.1
0.05	20.6	20.6	20.6
0.10	27.6		27.6
0.15	38.2	38.2	38.2
0.20	50.9		50.9
0.25	64.4	64.4	64.4
0.30	77.5		77.5
0.35	89.6	89.6	89.6
0.40	100.3		100.3
0.45	109.9	109.9	109.9
0.50	118.7		118.7
0.55	127.3	127.3	127.3
0.60	136.5	136.5	136.5
0.65	143.6		143.6
0.70	146.4	146.5	146.5
0.75	145.9	145.9	146.0
0.80	141.9		141.9
0.85	133.1	133.1	133.2
0.90	117.3		117.5
0.95	91.6	91.5	91.8
1.00	54.0	53.9	54.3
No. of function entries	80	60	

IV. DERIVATION OF FOURTH-ORDER RUNGE-KUTTA METHODS

FOR SOLUTION OF THE DIFFERENTIAL EQUATION

$$\frac{d^2 y}{dx^2} = f(x, y, y'); \quad y(x_0) = y_0; \quad y'(x_0) = u_0 \quad (4.0)$$

The classic derivation of the Runge-Kutta algorithm for a first-order equation was outlined briefly in Chapter II to indicate the manner in which the technique was originally formulated. The same pattern will be followed, in more detail, in building an analogous constraint matrix for methods directly applicable to solution of the swing equation.

In developing Taylor series expansions for functions of three variables

$$f(x_0 + \alpha_r, y_0 + \beta_r, u_0 + \gamma_r)$$

about the point (x_0, y_0, u_0) we use differential operators

$$D_r^j = \left(\alpha_r \frac{\partial}{\partial x} + \beta_r \frac{\partial}{\partial y} + \gamma_r \frac{\partial}{\partial u} \right)^j. \quad (4.1)$$

The expansions are then expressed as

$$f(x_0 + \alpha_r, y_0 + \beta_r, u_0 + \gamma_r) = f + D_r f + \frac{1}{2!} D_r^2 f + \dots \quad (4.2)$$

where

$$f = f(x_0, y_0, u_0).$$

Using the notation of (2.7ff), with addition of the subscript u such that

$$f_u = \frac{\partial}{\partial u} f,$$

we may expand the derivatives of $y(x_0)$.

$$y(x_0) = y_0$$

$$y'(x_0) = u_0 \quad (4.3)$$

$$y''(x_0) = y^{(2)}(x_0) = f \quad (4.4)$$

$$\begin{aligned} y^{(3)}(x_0) &= \frac{d}{dx} y^{(2)}(x_0) = \frac{df}{dx} \\ &= f_x + uf_y + ff_u = Df \end{aligned} \quad (4.5)$$

$$\begin{aligned} y^{(4)}(x_0) &= \frac{d}{dx} Df = f_{xx} + 2uf_{xy} + 2ff_{xu} + 2uff_{yu} \\ &\quad + u^2f_{yy} + f^2f_{uu} + f_u(f_x + uf_y + ff_u) + ff_y \end{aligned} \quad (4.6)$$

If we define

$$D^2f = f_{xx} + 2uf_{xy} + 2ff_{xu} + 2uff_{yu} + u^2f_{yy} + f^2f_{uu},$$

then

$$y^{(4)}(x_0) = D^2f + f_u Df + ff_y. \quad (4.7)$$

Similarly, defining

$$\begin{aligned} D^3f &= f_{xxx} + 3uf_{xxy} + 3ff_{xxu} + u^3f_{yyy} + 3u^2f_{xyy} \\ &\quad + 3u^2f_{yyu} + 6uff_{xyu} + 3f^2f_{xuu} + 3uf^2f_{yuu} \\ &\quad + f^3f_{uuu}, \end{aligned}$$

a tedious expansion of

$$\frac{d}{dx} y^{(4)}(x_0)$$

gives the expression

$$y^{(5)}(x_0) = D^3f + 3fDf_y + 3DfDf_u + f_u(D^2f + f_uDf + ff_y) + f_yDf. \quad (4.8)$$

The Taylor series for one variable gives

$$y(x_0 + h) = y(x_0) + hy'(x_0) + \frac{1}{2!} h^2 y''(x_0) + \frac{1}{3!} h^3 y^{(3)}(x_0) + \frac{1}{4!} h^4 y^{(4)}(x_0) + \frac{h^5}{5!} y^{(5)}(x_0) + O(h^6). \quad (4.9)$$

Although the object is to derive a fourth-order method, terms in h^5 will be retained to give an estimate of a good part of the truncation error.

Substitution of (4.3) - (4.8) into (4.9) gives the value of $y(x_0 + h)$ in terms of the function f .

$$\begin{aligned} y(x_0 + h) - y_0 &= hu_0 + \frac{h^2}{2} f + \frac{h^3}{6} (Df) \\ &+ \frac{h^4}{24} (D^2f + f_u Df + ff_y) \\ &+ \frac{h^5}{120} (D^3f + 3fDf_y + 3DfDf_u + f_u(D^2f + ff_y + f_u Df) + f_y Df) + O(h^6) \end{aligned} \quad (4.10)$$

Note, at this point, that terms in f are of the order of

$\frac{h^2}{2}$; a contrast from the first order derivation. This coincides with the results of Chapter III (see (3.19)). Hence we define the $\{k_i\}$ as follows:

$$k_1 = \frac{h^2}{2} f(x_0, y_0, u_0) = \frac{h^2}{2} f \quad (4.11)$$

$$k_2 = \frac{h^2}{2} f(x_0 + \alpha h, y_0 + \alpha h u_0 + \beta k_1, u_0 + \frac{2\gamma k_1}{h}), \quad (4.12)$$

$$k_3 = \frac{h^2}{2} f(x_0 + \alpha_1 h, y_0 + \alpha_1 h u_0 + \beta_1 k_1 + \delta_1 k_2, u_0 + \frac{2\gamma_1 k_1}{h} + \frac{2\epsilon_1 k_2}{h}), \quad (4.13)$$

$$k_4 = \frac{h^2}{2} f(x_0 + \alpha_2 h, y_0 + \alpha_2 h u_0 + \beta_2 k_1 + \delta_2 k_2 + \xi_2 k_3, u_0 + \frac{2}{h} (\gamma_2 k_1 + \epsilon_2 k_2 + \tau_2 k_3)), \quad (4.14)$$

$$k_5 = \frac{h^2}{2} f(x_0 + \alpha_3 h, y_0 + \alpha_3 h u_0 + \beta_3 k_1 + \delta_3 k_2 + \xi_3 k_3 + \lambda_3 k_4, u_0 + \frac{2}{h} (\gamma_3 k_1 + \epsilon_3 k_2 + \tau_3 k_3 + \phi_3 k_4)). \quad (4.15)$$

The expression for k_5 is carried along with this derivation for purposes of finding a method similar to Merson's for second-order equations.

The object of the derivation is to be able to express $[y(x_0 + h) - y(x_0)]$ and $[u(x_0 + h) - u(x_0)]$ in terms of (4.11) - (4.15), with suitable choice of $\{\alpha_i\}$ - $\{\phi_i\}$. The true value of $u(x_0 + h) - u(x_0)$ is given by the Taylor series expansion, as in (4.9):

$$u(x_0 + h) - u(x_0) = h u'(x_0) + \frac{h^2}{2!} u''(x_0) + \frac{h^3}{3!} u^{(3)}(x_0) + \frac{h^4}{4!} u^{(4)}(x_0) + \frac{h^5}{5!} u^{(5)}(x_0) + O(h^6) \quad (4.16)$$

Multiplying through this expression by h , and recalling that $u = \frac{dy}{dx}$, gives

$$\begin{aligned} hu(x_0 + h) = & hu_0 + 2 \frac{h^2}{2!} f + 3 \frac{h^3}{3!} (Df) + 4 \frac{h^4}{4!} (D^2f \\ & + f_u Df + f f_y) + 5 \frac{h^5}{5!} (D^3f + 3f Df_y + 3Df Df_u \\ & + f_u (D^2f + f_u Df + f f_y) + f_y Df) \\ & + O(h^6) = hu_1 \end{aligned} \quad (4.17)$$

Observing the form of (4.10) and (4.17), set

$$y_1 = y_0 + hu_0 + \sum_{j=1}^5 \gamma_{0j} k_j + O(h^6), \quad \text{and} \quad (4.18)$$

$$hu_1 = hu_0 + \sum_{j=1}^5 \gamma_{1j} k_j + O(h^6) \quad (4.19)$$

The $\{\gamma_{0j}\}$ and $\{\gamma_{1j}\}$ are constants appropriately chosen such that (4.18) and (4.19) correspond to (4.10) and (4.17) to order h^5 .

The $\{k_i\}$ must now be expanded using the Taylor series for three variables set forth in (4.1) and (4.2).

If it is assumed that

$$\gamma = \alpha \quad (4.20)$$

the Taylor's series for $\frac{2}{h^2} k_2$ becomes

$$\begin{aligned}
 \frac{2k_2}{h^2} = & f + (\alpha h) f_x + (\alpha h u) f_y + \frac{\beta h^2 f}{2} f_y + (\alpha h f) f_u \\
 & + \frac{1}{2!} \left[(\alpha^2 h^2) f_{xx} + (2\alpha^2 h^2 u) f_{yy} + 2\alpha^2 h^2 f f_{xu} + \right. \\
 & + (2\alpha^2 h^2 u f) f_{yu} + (\alpha^2 h^2 u^2) f_{yy} + (\alpha^2 h^2 f^2) f_{uu} + (\alpha \beta h^3 f) f_{xy} \\
 & + (\alpha \beta h^3 f^2) f_{yu} + (\alpha \beta h^3 u f) f_{yy} \left. \right] + \frac{1}{3!} \left[(\alpha^3 h^3) f_{xxx} \right. \\
 & + (3\alpha^3 h^3 u) f_{xxy} + (3\alpha^3 h^3 f) f_{xxu} + (6\alpha^3 h^3 u f) f_{xyu} \\
 & + (\alpha^3 h^3 u^3) f_{yyy} + (3\alpha^3 h^3 u^2) f_{xyy} + (3\alpha^3 h^3 u^2 f) f_{yyu} \\
 & + (3\alpha^3 h^3 u f^2) f_{yuu} + (3\alpha^3 h^3 f^2) f_{xuu} + (\alpha^3 h^3 f^3) f_{uuu} \left. \right] \\
 & + O(h^4).
 \end{aligned}$$

Collecting terms gives the final expression for k_2 ,

$$\begin{aligned}
 k_2 = & \frac{h^2}{2} (f) + \frac{h^3}{2} (\alpha Df) + \frac{h^4}{4} (\alpha^2 D^2 f + \beta f f_y) \\
 & + \frac{h^5}{12} (\alpha^3 D^3 f + 3\alpha \beta f Df_y) + O(h^6). \quad (4.21)
 \end{aligned}$$

Substitution of this value into (4.13) is necessary before

$\frac{2k_3}{h^2}$ can be expanded:

$$\begin{aligned}
 \frac{2k_3}{h^2} = & f(x_0 + \alpha_1 h, y_0 + \alpha_1 h u + \frac{(\beta_1 + \delta_1)}{2} h^2 f \\
 & + \frac{\alpha \delta_1 h^3}{2} Df, u_0 + \alpha_1 h f + \alpha \epsilon_1 h^2 Df \\
 & + \frac{h^3}{2} (\alpha^2 \epsilon_1 D^2 f + \beta \epsilon_1 f f_y)).
 \end{aligned}$$

If it is assumed that

$$\gamma_1 + \epsilon_1 = \alpha_1 \quad (4.22)$$

this becomes

$$\begin{aligned} \frac{2k_3}{h^2} = & f + \alpha_1 h Df + \frac{\beta_1 + \delta_1}{2} h^2 f f_y + \alpha \epsilon_1 h^2 f_u Df + \frac{\alpha \delta_1 h^3}{2} f_y Df \\ & + \frac{\alpha^2 \epsilon_1 h^3}{2} f_u D^2 f + \frac{\beta \epsilon_1 h^3}{2} f f_y f_u + \frac{1}{2!} (\alpha_1^2 h^2 D^2 f \\ & + \frac{2\alpha_1 (\beta_1 + \delta_1)}{2} h^3 f f_{xy} + \alpha_1 (\beta_1 + \delta_1) h^3 f_{yy} \\ & + 2\alpha \alpha_1 \epsilon_1 h^3 f_{xy} Df + 2\alpha \alpha_1 \epsilon_1 h^3 u f_{yu} Df + 2\alpha \alpha_1 \epsilon_1 h^3 f f_{uu} Df \\ & + \alpha_1 (\beta_1 + \delta_1) h^3 f f_{yu}) + \frac{1}{3!} (\alpha_1^3 h^3 D^3 f) + O(h^4). \end{aligned}$$

Collecting terms gives the final expression for k_3 :

$$\begin{aligned} k_3 = & \frac{h^2}{2} (f) + \frac{h^3}{2} (\alpha_1 Df) + \frac{h^4}{4} (\alpha_1^2 D^2 f + (\beta_1 + \delta_1) f f_y \\ & + 2\alpha \epsilon_1 f_u Df) + \frac{h^5}{12} (\alpha_1^3 D^3 f + 3\alpha \delta_1 f_y Df + 3\alpha^2 \epsilon_1 f_u D^2 f \\ & + 3\beta \epsilon_1 f f_y f_u + 3\alpha_1 (\beta_1 + \delta_1) f Df_y + 6\alpha \alpha_1 \epsilon_1 Df Df_u) \\ & + O(h^6). \end{aligned} \quad (4.23)$$

Similar operations yield the expansion of k_4 and k_5 .

If

$$\gamma_2 + \epsilon_2 + \tau_2 = \alpha_2, \text{ then} \quad (4.24)$$

$$\begin{aligned}
 k_4 = & \frac{h^2}{2} (f) + \frac{h^3}{2} (\alpha_2 Df) + \frac{h^4}{4} (\alpha_2^2 D^2 f + (\beta_2 + \delta_2 + \xi_2) f f_y \\
 & + 2(\alpha \epsilon_2 + \alpha_1 \tau_2) f_u Df) + \frac{h^5}{12} (\alpha_2^3 D^3 f + 3(\alpha \delta_2 + \alpha_1 \xi_2) f_y Df \\
 & + 3(\alpha^2 \epsilon_2 + \alpha_1^2 \tau_2) f_u D^2 f + 3(\beta \epsilon_2 + \beta_1 \tau_2 + \delta_1 \tau_2) f f_y f_u \\
 & + 3\alpha_2 (\beta_2 + \delta_2 + \xi_2) f Df_y + 6\alpha \epsilon_1 \tau_2 f_u^2 Df \\
 & + 6\alpha_2 (\alpha \epsilon_2 + \alpha_1 \tau_2) Df Df_u) + O(h^6); \quad (4.25)
 \end{aligned}$$

and if $\gamma_3 + \epsilon_3 + \tau_3 + \phi_3 = \alpha_3$, then

$$\begin{aligned}
 k_5 = & \frac{h^2}{2} (f) + \frac{h^3}{2} (\alpha_3 Df) + \frac{h^4}{4} (\alpha_3^2 D^2 f + (\beta_3 + \delta_3 + \lambda_3 \\
 & + \xi_3) f f_y + 2(\alpha \epsilon_3 + \alpha_1 \tau_3 + \alpha_2 \phi_3) f_u Df) + \frac{h^5}{12} (\alpha_3^3 D^3 f \\
 & + 3(\alpha \delta_3 + \alpha_1 \xi_3 + \alpha_2 \lambda_3) f_y Df + 3(\alpha^2 \epsilon_3 + \alpha_1^2 \tau_3 + \\
 & + \alpha_2^2 \phi_3) f_u D^2 f + 3(\beta \epsilon_3 + (\beta_1 + \delta_1) \tau_3 + (\beta_2 + \delta_2 \\
 & + \xi_2) \phi_3) f f_y f_u + 6(\alpha \epsilon_1 \tau_3 + (\alpha \epsilon_2 + \alpha_1 \tau_2) \phi_3) f_u^2 Df \\
 & + 3\alpha_3 (\beta_3 + \delta_3 + \xi_3 + \lambda_3) f Df_y + 6\alpha_3 (\alpha \epsilon_3 + \alpha_1 \tau_3 \\
 & + \alpha_2 \phi_3) Df Df_u) + O(h^6).
 \end{aligned}$$

These results may now be compiled to give the constraints for the second-order equation Runge-Kutta algorithm. Until needed, terms in k_5 will be deleted from the compilation.

For correspondence between terms of (4.18) and those of (4.10) to an order of h^4 in $y(x_0 + h)$, we have:

$$\text{from } \frac{h^2}{2} f : \gamma_{01} + \gamma_{02} + \gamma_{03} + \gamma_{04} = 1 \quad (4.27)$$

$$\text{from } \frac{h^3}{2} Df : \alpha \gamma_{02} + \alpha_1 \gamma_{03} + \alpha_2 \gamma_{04} = \frac{1}{3} \quad (4.28)$$

$$\text{from } \frac{h^4}{4} D^2 f : \alpha^2 \gamma_{02} + \alpha_1 \gamma_{03} + \alpha_2^2 \gamma_{04} = \frac{1}{6} \quad (4.29)$$

$$\text{from } \frac{h^4}{4} f f_y : \beta \gamma_{02} + (\beta_1 + \delta_1) \gamma_{03} + (\beta_2 + \delta_2 + \xi_2) \gamma_{04} = \frac{1}{6} \quad (4.30)$$

$$\text{from } \frac{h^4}{4} f_u Df : 2\alpha \epsilon_1 \gamma_{03} + 2(\alpha \epsilon_2 + \alpha_1 \tau_2) \gamma_{04} = \frac{1}{6} \quad (4.31)$$

For fifth-order correspondence, the following relationships must also hold:

$$\text{from } \frac{h^5}{12} D^3 f : \alpha^3 \gamma_{02} + \alpha_1^3 \gamma_{03} + \alpha_2^3 \gamma_{04} = \frac{1}{10} \quad (4.32)$$

$$\text{from } \frac{h^5}{4} D^3 f : \alpha \beta \gamma_{02} + \alpha_1 (\beta_1 + \delta_1) \gamma_{03} + \alpha_2 (\beta_2 + \delta_2 + \xi_2) \gamma_{04} = \frac{1}{10} \quad (4.33)$$

$$\text{from } \frac{h^5}{4} Df Df_u : 2\alpha \alpha_1 \epsilon_1 \gamma_{03} + 2\alpha_2 (\alpha \epsilon_2 + \alpha_1 \tau_2) \gamma_{04} = \frac{1}{10} \quad (4.34)$$

$$\text{from } \frac{h^5}{4} f_y Df : \alpha \delta_1 \gamma_{03} + (\alpha \delta_2 + \alpha_1 \xi_2) \gamma_{04} = \frac{1}{30} \quad (4.35)$$

$$\text{from } \frac{h^5}{4} f_u D^2 f : \alpha^2 \epsilon_1 \gamma_{03} + (\alpha^2 \epsilon_2 + \alpha_1^2 \tau_2) \gamma_{04} = \frac{1}{30} \quad (4.36)$$

$$\text{from } \frac{h^5}{4} f f_y f_u : \beta \epsilon_1 \gamma_{03} + (\beta \epsilon_2 + (\beta_1 + \delta_1) \tau_2) \gamma_{04} = \frac{1}{30} \quad (4.37)$$

$$\text{from } \frac{h^5}{4} f_u^2 Df : \alpha \epsilon_1 \tau_2 \gamma_{04} = \frac{1}{60} \quad (4.38)$$

Likewise, for terms in (4.19) to correspond to those in (4.17) through order h^5 ,

$$\text{from } \frac{h^2}{2} f : \gamma_{11} + \gamma_{12} + \gamma_{13} + \gamma_{14} = 2 \quad (4.39)$$

$$\text{from } \frac{h^3}{2} Df : \alpha \gamma_{12} + \alpha_1 \gamma_{13} + \alpha_2 \gamma_{14} = 1 \quad (4.40)$$

$$\text{from } \frac{h^4}{4} D^2 f : \alpha^2 \gamma_{12} + \alpha_1^2 \gamma_{13} + \alpha_2^2 \gamma_{14} = \frac{2}{3} \quad (4.41)$$

$$\text{from } \frac{h^4}{4} ff_y : \beta \gamma_{12} + (\beta_1 + \delta_1) \gamma_{13} + (\beta_2 + \delta_2 + \xi_2) \gamma_{14} = \frac{2}{3} \quad (4.42)$$

$$\text{from } \frac{h^4}{4} f_u Df : 2\alpha \epsilon_1 \gamma_{13} + 2(\alpha \epsilon_2 + \alpha_1 \tau_2) \gamma_{14} = \frac{2}{3} \quad (4.43)$$

$$\text{from } \frac{h^5}{12} D^3 f : \alpha^3 \gamma_{12} + \alpha_1^3 \gamma_{13} + \alpha_2^3 \gamma_{14} = \frac{1}{2} \quad (4.44)$$

$$\text{from } \frac{h^5}{4} f Df_y : \alpha \beta \gamma_{12} + \alpha_1 (\beta_1 + \delta_1) \gamma_{13} + \alpha_2 (\beta_2 + \delta_2 + \xi_2) \gamma_{14} = \frac{1}{2} \quad (4.45)$$

$$\text{from } \frac{h^5}{4} Df Df_u : 2\alpha \alpha_1 \epsilon_1 \gamma_{13} + 2\alpha_2 (\alpha \epsilon_2 + \alpha_1 \tau_2) \gamma_{14} = \frac{1}{2} \quad (4.46)$$

$$\text{from } \frac{h^5}{4} f_y Df : \alpha \delta_1 \gamma_{13} + (\alpha \delta_2 + \alpha_1 \xi_2) \gamma_{14} = \frac{1}{6} \quad (4.47)$$

$$\text{from } \frac{h^5}{4} f_u D^2 f : \alpha^2 \epsilon_1 \gamma_{13} + (\alpha^2 \epsilon_2 + \alpha_1^2 \tau_2) \gamma_{14} = \frac{1}{6} \quad (4.48)$$

$$\text{from } \frac{h^5}{4} ff_y f_u : \beta \epsilon_1 \gamma_{13} + (\beta \epsilon_2 + (\beta_1 + \delta_1) \tau_2) \gamma_{14} = \frac{1}{6} \quad (4.49)$$

$$\text{from } \frac{h^5}{2} f_u^2 Df : \alpha \epsilon_1 \tau_2 \gamma_{14} = \frac{1}{12} \quad (4.50)$$

In matrix form, (4.27) to (4.50) are:

$$\begin{bmatrix}
 1 & 1 & 1 & 1 \\
 0 & \alpha & \alpha_1 & \alpha_2 \\
 0 & \alpha^2 & \alpha_1^2 & \alpha_2^2 \\
 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \xi_2 \\
 0 & 0 & 2\alpha\epsilon_1 & 2(\alpha\epsilon_2 + \alpha_1\tau_2) \\
 \dots & \dots & \dots & \dots \\
 0 & \alpha^3 & \alpha_1^3 & \alpha_2^3 \\
 0 & \alpha\beta & \alpha_1(\beta_1 + \delta_1) & \alpha_2(\beta_2 + \delta_2 + \xi_2) \\
 0 & 0 & 2\alpha\alpha_1\epsilon_1 & 2\alpha_2(\alpha\epsilon_2 + \alpha_1\tau_2) \\
 0 & 0 & \alpha\delta_1 & \alpha\delta_2 + \alpha_1\xi_2 \\
 0 & 0 & \alpha^2\epsilon_1 & \alpha^2\epsilon_2 + \alpha_1^2\tau_2 \\
 0 & 0 & \beta\epsilon_1 & \beta\epsilon_2 + (\beta_1 + \delta_1)\tau_2 \\
 0 & 0 & 0 & \alpha\epsilon_1\tau_2
 \end{bmatrix} \cdot \begin{bmatrix} \gamma_{01} & \gamma_{11} \\ \gamma_{02} & \gamma_{12} \\ \gamma_{03} & \gamma_{13} \\ \gamma_{04} & \gamma_{14} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \dots & \dots \\ \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{60} & \frac{1}{12}
 \end{bmatrix} \quad (4.51)$$

Provided the relations set forth in the upper submatrix are satisfied, and provided

$$\alpha = \gamma ;$$

$$\alpha_1 = \gamma_1 + \epsilon_1 ; \text{ and} \quad (4.52)$$

$$\alpha_2 = \gamma_2 + \epsilon_2 + \tau_2 ,$$

the following procedure is a Runge-Kutta method of fourth-order accuracy in h , suitable for incremental solution of the swing equation as set forth in (4.0):

x_i	y_i	$v_i = hy'_i = hu_i$	k_i
x_0	y_0	v_0	k_1
$x_0 + \alpha h$	$y_0 + \alpha v_0 + \beta k_1$	$v_0 + 2\gamma k_1$	k_2
$x_0 + \alpha_1 h$	$y_0 + \alpha_1 v_0 + \beta_1 k_1 + \delta_1 k_2$	$v_0 + 2\gamma_1 k_1 + 2\epsilon_1 k_2$	k_3
$x_0 + \alpha_2 h$	$y_0 + \alpha_2 v_0 + \beta_2 k_1 + \delta_2 k_2 + \xi_2 k_3$	$v_0 + 2\gamma_2 k_1 + 2\epsilon_2 k_2 + 2\tau_2 k_3$	k_4
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + \sum_{j=1}^4 \gamma_{0j} k_j$	$v_1 = v_0 + \sum_{j=1}^4 \gamma_{1j} k_j$	

In each case $k_i = \frac{h^2}{2} f(x_i, y_i, u_i) . \quad (4.53)$

V. SELECTION OF THE RUNGE-KUTTA SECOND-ORDER
METHOD MOST SUITABLE FOR DIGITAL COMPUTER
SOLUTION OF THE SWING EQUATION

With (4.51) - (4.53) in hand, it is possible to synthesize an endless number of numerical integration techniques, all of which are appropriate to the general equation

$$\frac{d^2 y}{dx^2} = f(x, y, u) ; \left(u = \frac{dy}{dx} \right); \text{ with initial conditions.} \quad (5.1)$$

As an illustration of the procedure to be followed, the technique introduced by Johnson and Ward in "The Solution of Power System Stability Problems by Means of Digital Computers",⁽²⁾ will be developed.

5.1 Development of the Johnson-Ward Formulae

Maintaining the notation of Chapter IV, set

$$\begin{aligned} \alpha &= \frac{1}{2} , & \gamma &= \frac{1}{2} ; & \gamma_{11} &= \frac{1}{3} ; \\ \alpha_1 &= \frac{1}{2} , & \gamma_1 &= 0 ; & \epsilon_1 &= \frac{1}{2} ; & \gamma_{12} &= \frac{2}{3} ; \\ \alpha_2 &= 1 , & \gamma_2 &= 0 ; & \epsilon_2 &= 0 ; & \tau_2 &= 1 ; & \gamma_{13} &= \frac{2}{3} ; & \gamma_{14} &= \frac{1}{3} \end{aligned} \quad (5.2)$$

This is in accordance with the original procedure of Kutta: the function is evaluated at the beginning of the step interval, twice at the mid-point of the interval, and once at the end.

The constraint matrix for the above values, from

(4.51), becomes that of (5.3).

$$\begin{bmatrix}
 1 & 1 & 1 & 1 \\
 0 & \frac{1}{2} & \frac{1}{2} & 1 \\
 0 & \frac{1}{4} & \frac{1}{4} & 1 \\
 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \xi_2 \\
 \dots & \dots & \dots & \dots \\
 0 & \frac{1}{8} & \frac{1}{8} & 1 \\
 0 & \frac{\beta}{2} & \frac{\beta_1 + \delta_1}{2} & \beta_2 + \delta_2 + \xi_2 \\
 0 & 0 & \frac{1}{4} & 1 \\
 0 & 0 & \frac{\delta_1}{2} & \frac{\delta_2 + \xi_2}{2} \\
 0 & 0 & \frac{1}{8} & \frac{1}{4} \\
 0 & 0 & \frac{\beta}{2} & \beta_1 + \delta_1 \\
 0 & 0 & 0 & \frac{1}{4}
 \end{bmatrix}
 \begin{bmatrix}
 \gamma_{01} & \frac{1}{3} \\
 \gamma_{02} & \frac{2}{3} \\
 \gamma_{03} & \frac{2}{3} \\
 \gamma_{04} & \frac{1}{3}
 \end{bmatrix}
 =
 \begin{bmatrix}
 1 & 2 \\
 \frac{1}{3} & 1 \\
 \frac{1}{6} & \frac{2}{3} \\
 \frac{1}{6} & \frac{2}{3} \\
 \dots & \dots \\
 - & \frac{1}{2} \\
 - & - \\
 - & \frac{1}{2} \\
 - & - \\
 - & \frac{1}{6} \\
 - & - \\
 - & \frac{1}{12}
 \end{bmatrix}
 \quad (5.3)$$

From this information, noting that the equations expressed by the upper sub-matrix must hold true to give accuracy of $O(h^4)$,

$$\frac{2}{3} \beta + \frac{2}{3}(\beta_1 + \delta_1) + \frac{1}{3}(\beta_2 + \delta_2 + \xi_2) = \frac{2}{3} \quad (5.4)$$

$$\gamma_{01} + \gamma_{02} + \gamma_{03} + \gamma_{04} = 1 \quad (5.5)$$

$$\frac{1}{2} \gamma_{02} + \frac{1}{2} \gamma_{03} + \gamma_{04} = \frac{1}{3} \quad (5.6)$$

$$\frac{1}{4} \gamma_{02} + \frac{1}{4} \gamma_{03} + \gamma_{04} = \frac{1}{6} \quad (5.7)$$

$$\frac{1}{2} \gamma_{03} + \gamma_{04} = \frac{1}{6} \quad (5.8)$$

$$\text{From (5.6) and (5.7), } \gamma_{04} = 0; \quad \gamma_{02} + \gamma_{03} = \frac{2}{3} . \quad (5.9)$$

$$\text{From (5.8), } \gamma_{03} = \frac{1}{3} .$$

$$\text{Hence, } \gamma_{02} = \frac{1}{3} \text{ and, from (5.5), } \gamma_{01} = \frac{1}{3} .$$

We can then write

$$\frac{1}{3} \beta + \frac{1}{3} (\beta_1 + \delta_1) = \frac{1}{6} \quad (5.10)$$

and from (5.4) it follows that

$$\beta_2 + \delta_2 + \xi_2 = 1 .$$

$$\text{Let us choose } \beta = \frac{1}{4} , \quad \delta_1 = \frac{1}{4} \text{ and } \xi_2 = 1 .$$

$$\text{Then } \beta_1 + \delta_1 = \frac{1}{4} \text{ so } \beta_1 = 0; \text{ also } \beta_2 = \delta_2 = 0 .$$

The final constraint matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 1/4 & 1/4 & 1 \\ 0 & 1/4 & 1/4 & 1 \\ 0 & 0 & 1/2 & 1 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 1/8 & 1/8 & 1 \\ 0 & 1/8 & 1/8 & 1 \\ 0 & 1/8 & 1/4 & 1 \\ 0 & 0 & 1/8 & 1/2 \\ 0 & 0 & 1/8 & 1/4 \\ 0 & 0 & 1/8 & 1/4 \\ 0 & 0 & 0 & 1/4 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & 2/3 \\ 1/3 & 2/3 \\ 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1/3 & 1 \\ 1/6 & 2/3 \\ 1/6 & 2/3 \\ 1/6 & 2/3 \\ \cdot & \cdot \\ 1/12 & 1/2 \\ 1/12 & 1/2 \\ 1/12 & 1/2 \\ 1/24 & 1/4 \\ 1/24 & 1/6 \\ 1/24 & 1/6 \\ 0 & 1/12 \end{bmatrix} \quad (5.11)$$

and the net procedure is set forth in (5.12) —

x_i	y_i	$v_i = hu_i$	$k_i = \frac{h^2}{2} f(x_i, y_i, u_i)$
x_0	y_0	v_0	k_1
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2} v_0 + \frac{1}{4} k_1$	$v_0 + k_1$	k_2
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2} v_0 + \frac{1}{4} k_2$	$v_0 + k_2$	k_3
$x_0 + h$	$y_0 + v_0 + k_3$	$v_0 + 2k_3$	k_4
			$k = \frac{1}{3}(k_1 + k_2 + k_3)$
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k'$	$k' = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$

(5.12)

Since the right-hand side of the upper sub-matrix in (5.11) corresponds to that in (4.51) the truncation in y and v is of order h^4 . To obtain and estimate of the truncation error, the lower sub-matrices of the right-hand sides of these equations must be examined. Let TR represent the sub-matrix from (4.51) and JW represent that from (5.11). Then

$$\underline{\text{TR}} = \begin{bmatrix} \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} \\ \frac{1}{10} & \frac{1}{2} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{30} & \frac{1}{6} \\ \frac{1}{60} & \frac{1}{12} \end{bmatrix} \quad \text{and} \quad \underline{\text{JW}} = \begin{bmatrix} \frac{1}{12} & \frac{1}{2} \\ \frac{1}{12} & \frac{1}{2} \\ \frac{1}{12} & \frac{1}{2} \\ \frac{1}{24} & \frac{1}{4} \\ \frac{1}{24} & \frac{1}{6} \\ \frac{1}{24} & \frac{1}{6} \\ 0 & \frac{1}{2} \end{bmatrix} \begin{matrix} \cdot \cdot \cdot \text{ row a} \\ \cdot \cdot \cdot \text{ row b} \\ \cdot \cdot \cdot \text{ row c} \\ \cdot \cdot \cdot \text{ row d} \\ \cdot \cdot \cdot \text{ row e} \\ \cdot \cdot \cdot \text{ row f} \\ \cdot \cdot \cdot \text{ row g} \end{matrix} \quad (5.13)$$

To evaluate the error of order h^5 , first of all recall that if each row in JW is identical to the corresponding row in TR there is no truncation error of $O(h^5)$ in the process JW represents. Also recall that the first column of these submatrices refers to truncation of y_i ; the second column to truncation of $v_i = hy_i'$. If we set

$$\begin{aligned}
 a &= \text{error component due to term } \frac{h^5}{12} D^3 f \\
 b &= \text{error component due to term } \frac{h^5}{4} f D f_y \\
 c &= \text{error component due to term } \frac{h^5}{4} D f D f_u \\
 d &= \text{error component due to term } \frac{h^5}{4} f_y D f \\
 e &= \text{error component due to term } \frac{h^5}{4} f_u D^2 f \\
 f &= \text{error component due to term } \frac{h^5}{4} f f_y f_u \\
 g &= \text{error component due to term } \frac{h^5}{2} f_u^2 D f ,
 \end{aligned} \tag{5.14}$$

recalling the terms from which the matrix was derived, we can express the truncation error components as in (5.15) and (5.16):

$$\begin{aligned}
 \epsilon(y_1 - y_0) &= -\frac{1}{60} a - \frac{1}{60} b - \frac{1}{60} c + \frac{1}{120} e \\
 &\quad + \frac{1}{120} f - \frac{1}{60} g + O(h^6)
 \end{aligned} \tag{5.15}$$

and

$$\epsilon(hy_1' - hy_0') = \frac{1}{12} d + O(h^6) . \tag{5.16}$$

From (5.16) it is apparent that the truncation error in y_i' is not of order h^5 , but of order h^4 . This is poor for our purposes, as shall be shown. The process requires a very small value of h to give satisfactory results.

Another, more accurate procedure for second-order differential equations is given by Collatz⁽⁸⁾. To the author's knowledge this has not been previously applied to the swing equation, yet it is more accurate than the method set forth by Johnson and Ward, and requires less computing time when applied to a function which does not explicitly contain $y'(x)$, such as the swing equation in its most commonly-used form (with no damping term). The Collatz procedure is set forth below:

x_i	y_i	$v_i = hy_i'$	$k_i = \frac{h^2}{2} f(x_i, y_i, y_i')$
x_0	y_0	v_0	k_1
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2} v_0 + \frac{1}{4} k_1$	$v_0 + k_1$	k_2
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2} v_0 + \frac{1}{4} k_1$	$v_0 + k_2$	k_3
$x_0 + h$	$y_0 + v_0 + k_3$	$v_0 + 2k_3$	k_4
			$k = \frac{1}{3}(k_1 + k_2 + k_3)$
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k_1$	$k' = \frac{1}{3}(k_1 + 2k_2 + 2k_3 + k_4)$

(5.17)

The error expressions for the Collatz procedure are

$$\begin{aligned}\epsilon(y_1 - y_0) &= -\frac{1}{60}a - \frac{1}{60}b - \frac{1}{60}c - \frac{1}{30}d + \frac{1}{120}f \\ &\quad - \frac{1}{60}g + O(h^6)\end{aligned}\tag{5.18}$$

and

$$\epsilon(hy_1' - hy_0') = O(h^6)\tag{5.16}$$

Note that here $\epsilon(y_1' - y_0') = O(h^5)$, whereas in the Johnson-Ward procedure $\epsilon(y_1' - y_0') = O(h^4)$.

Example 5.1

The two methods are compared by a numerical integration of the swing equation of Example 3.1, assuming a fault clearing time of 0.60 second. Computer results for various values of step length, h , are given in Appendix (C). Figure 5.1 shows the swing curves generated by each method for $h = 0.20$ and $h = 0.10$ second.

The Collatz method is clearly the superior in this application; it requires just three function entries per step because $k_2 = k_3$ in this case. The other procedure, in addition to requiring much more computer time to solve the problem, is far less accurate.

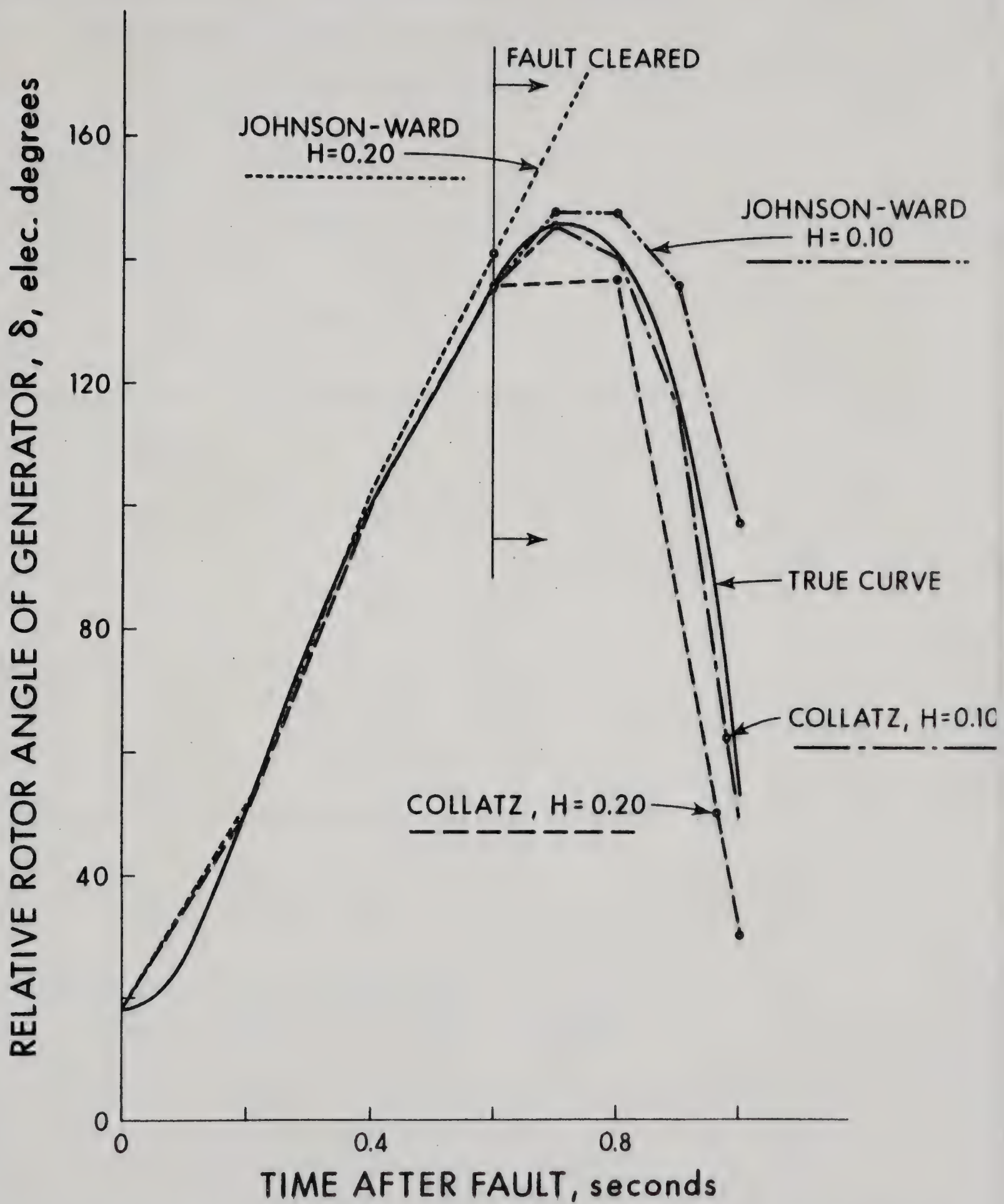


FIG. 5.1 SWING CURVES OF EXAMPLE 5.1 - CLEAR. TIME 0.60 SEC.

5.2 Formulation of New Second Order Runge-Kutta Procedures Pattern After Common First Order Procedures

A. A Method Based on Strachey, Eqn. (3.24)

The first-order Strachey process for solution of

$$\frac{dy}{dx} = f(x, y)$$

is characterized by the following integration scheme:

$$\begin{aligned} k_1 &= hf(x_0, y_0) \\ k_2 &= hf(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1) \\ k_3 &= hf(x_0 + \frac{1}{2}h, y_0 - \frac{1}{2}k_1 + \frac{1}{2}k_2) \\ k_4 &= hf(x_0 + h, y_0 + \frac{1}{2}k_2 + \frac{1}{2}k_3) \\ y_1 &= y_0 + \frac{1}{6}k_1 + \frac{1}{2}k_2 + \frac{1}{6}k_3 + \frac{1}{6}k_4 \end{aligned} \tag{5.20}$$

When forming a second-order method, for solution of

$$\frac{d^2y}{dx^2} = f(x, y, y')$$

let us increment dy/dx in accordance with the first-order method. Using the table of (4.53), this gives

$$\begin{array}{llll} \gamma_{11} = \frac{1}{3} & & & \\ \gamma_{12} = 1 & \alpha = \frac{1}{2} & \gamma = \frac{1}{2} & \\ \gamma_{13} = \frac{1}{3} & \alpha_1 = \frac{1}{2} & \gamma_1 = -\frac{1}{2} & \epsilon_1 = 1 \\ \gamma_{14} = \frac{1}{3} & \alpha_2 = 1 & \gamma_2 = 0 & \epsilon_2 = \frac{1}{2} \quad \tau_2 = \frac{1}{2} \end{array}$$

Note that constraints (4.52) are satisfied, since

$$\begin{aligned}\gamma &= \alpha, \\ \gamma_1 + \epsilon_1 &= \alpha_1, \quad \text{and} \\ \gamma_2 + \epsilon_2 + \tau_2 &= \alpha_2.\end{aligned}$$

Substitution of these values into (4.51) permits formation of the constraint matrices:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \xi_2 \\ 0 & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & \frac{1}{8} & \frac{1}{8} & 1 \\ 0 & \frac{\beta}{2} & \frac{\beta_1 + \delta_1}{2} & \beta_2 + \delta_2 + \xi_2 \\ 0 & 0 & \frac{1}{2} & 1 \\ 0 & 0 & \frac{\delta_1}{2} & \frac{\delta_2 + \xi_2}{2} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \beta & \frac{\beta}{2} + \frac{\beta_1 + \delta_1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} \gamma_{01} & \frac{1}{3} \\ \gamma_{02} & 1 \\ \gamma_{03} & \frac{1}{3} \\ \gamma_{04} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \cdot & \cdot \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \end{bmatrix} \quad (5.21)$$

From (5.21) we may set forth the equations that must hold for accuracy of h^4 in $(y_1 - y_0)$ and $(y_1' - y_0')$:

$$\gamma_{01} + \gamma_{02} + \gamma_{03} + \gamma_{04} = 1$$

$$\frac{1}{2} \gamma_{02} + \frac{1}{2} \gamma_{03} + \gamma_{04} = \frac{1}{3}$$

$$\frac{1}{4} \gamma_{02} + \frac{1}{4} \gamma_{03} + \gamma_{04} = \frac{1}{6}$$

$$\gamma_{03} + \gamma_{04} = \frac{1}{6}$$

$$\beta + \frac{1}{3}(\beta_1 + \delta_1) + \frac{1}{3}(\beta_2 + \delta_2 + \xi_2) = \frac{2}{3}$$

$$\frac{1}{2} \beta + \frac{1}{6}(\beta_1 + \delta_1) = \frac{1}{6}$$

Also,

$$\begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \\ \gamma_{04} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \\ 0 \end{bmatrix} \quad (5.22)$$

and $(\beta_2 + \delta_2 + \xi_2) = 1$.

From (5.21), using (5.22), $\delta_1 = \frac{1}{3}$.

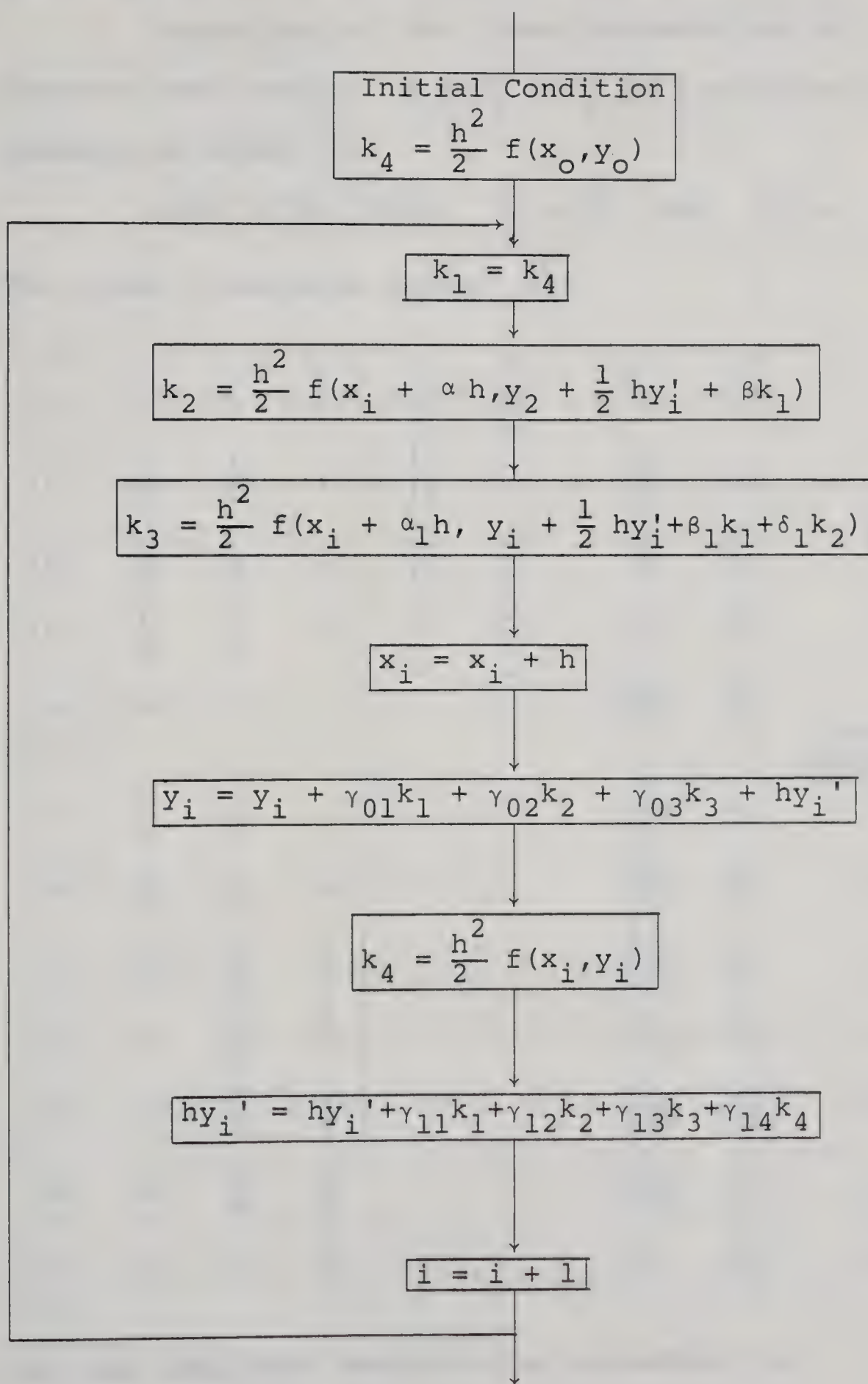
Note now that rapidity of integration will be achieved if we set

$$\begin{bmatrix} \beta_2 \\ \delta_2 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{2} \\ \frac{1}{6} \end{bmatrix} \quad (5.23)$$

Assuming we are dealing with the equation

$$\frac{d^2 y}{dx^2} = f(x, y),$$

where y' does not appear explicitly, the integration may then be accomplished in the following manner:



Note that only three function entries are required per iteration, as compared to the usual four.

Inspection of the lower sub-matrices of (5.19) indicates that setting $\beta = \beta_1 + \delta_1$ will minimize error components of order h^5 .

From (5.2), then, $\beta = \frac{1}{4}$ and $\beta_1 = -\frac{1}{12}$.

The final constraint matrix is

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & 0 & 1 & 1 \\ 0 & \frac{1}{8} & \frac{1}{8} & 1 \\ 0 & \frac{1}{8} & \frac{1}{8} & 1 \\ 0 & 0 & \frac{1}{2} & \frac{1}{1} \\ 0 & 0 & \frac{1}{6} & \frac{1}{3} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 \\ \frac{1}{6} & \frac{1}{3} \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{12} & \frac{1}{2} \\ \frac{1}{12} & \frac{1}{2} \\ \frac{1}{12} & \frac{1}{2} \\ \frac{1}{36} & \frac{1}{6} \\ \frac{1}{24} & \frac{1}{6} \\ \frac{1}{24} & \frac{1}{6} \\ 0 & \frac{1}{6} \end{bmatrix} \begin{array}{c} \text{Error} \\ \text{Components} \\ a \\ b \\ c \\ d \\ e \\ f \\ g \end{array} \quad (5.24)$$

and the completed second-order procedure is:

x_i	y_i	$v_i = hy_i'$	$k_i = \frac{h^2}{2} f(x_i, y_i, y_i')$
x_0	y_0	v_0	k_1
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2}v_0 + \frac{1}{4}k_1$	$v_0 + k_1$	k_2
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2}v_0 - \frac{1}{12}k_1 + \frac{1}{3}k_2$	$v_0 - k_1 + 2k_2$	k_3
$x_0 + h$	$y_0 + v_0 + \frac{1}{6}(2k_1 + 3k_2 + k_3)$	$v_0 + k_2 + k_3$	k_4
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k'$	$k = \frac{1}{3}k_1 + \frac{1}{2}k_2 + \frac{1}{6}k_3$ $k' = \frac{1}{3}(k_1 + 3k_2 + k_3 + k_4)$

(5.25)

In a manner identical to that of the previous section, we may determine the error introduced that is of the order of h^5 .

$$\begin{aligned} \epsilon(y_1 - y_0) &= -\frac{1}{60}a - \frac{1}{60}b - \frac{1}{60}c - \frac{1}{180}d + \frac{1}{120}e \\ &\quad + \frac{1}{120}f - \frac{1}{60}g + O(h^6) \end{aligned} \quad (5.26)$$

$$\epsilon(hy_1' - hy_0') = \frac{1}{12}g + O(h^6) \quad (5.27)$$

Unfortunately, the truncation error in $(y_1' - y_0')$ here is of $O(h^4)$, as it was in the Johnson-Ward procedure. This error term was inherited directly from the Strachey procedure, rather than from the truncation of y_i . However, the error terms are slightly smaller in $O(h^5)$ than are their Johnson-Ward equivalents, so it could reasonably be expected that this process is more accurate.

B. A Method Based on Gill, Eqn. (3.22)

In a manner similar to that of the previous section, we increment dy/dx in accordance with the Gill procedure for first-order equations. Again, using the table of (4.53), we obtain

$$\begin{aligned} \gamma_{11} &= \frac{1}{3} \\ \gamma_{12} &= \frac{2}{3} \left(1 - \frac{1}{\sqrt{2}}\right) \quad \alpha = \frac{1}{2} \quad \gamma = \frac{1}{2} \\ \gamma_{13} &= \frac{2}{3} \left(1 + \frac{1}{\sqrt{2}}\right) \quad \alpha_1 = \frac{1}{2} \quad \gamma_1 = -\frac{1}{2} + \frac{1}{\sqrt{2}} \quad \epsilon_1 = 1 - \frac{1}{\sqrt{2}} \\ \gamma_{14} &= \frac{1}{3} \quad \alpha_2 = 1 \quad \gamma_2 = 0 \quad \epsilon_2 = -\frac{1}{\sqrt{2}} \quad \tau_2 = 1 + \frac{1}{\sqrt{2}} . \end{aligned} \quad (5.28)$$

Note that constraints (4.52) are satisfied.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & \frac{1}{4} & \frac{1}{4} & 1 \\ 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \epsilon_2 \\ 0 & 0 & 1 - \frac{1}{\sqrt{2}} & 1 \\ 0 & \frac{1}{8} & \frac{1}{8} & 1 \\ 0 & \frac{\beta}{2} & \frac{\beta_1 + \delta_1}{2} & \beta_2 + \delta_2 + \epsilon_2 \\ 0 & 0 & \frac{1}{2} \left(1 - \frac{1}{\sqrt{2}}\right) & 1 \\ 0 & 0 & \frac{\delta_1}{2} & \frac{\delta_2 + \epsilon_2}{2} \\ 0 & 0 & \frac{1}{4} \left(1 - \frac{1}{\sqrt{2}}\right) & \frac{1}{4} \\ 0 & 0 & \left(1 - \frac{1}{\sqrt{2}}\right) & \frac{\beta_1 + \delta_1 - \beta}{\sqrt{2}} + (\beta_1 + \delta_1) \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} \gamma_{01} & \frac{1}{3} \\ \gamma_{02} & \frac{2}{3} \left(1 - \frac{1}{\sqrt{2}}\right) \\ \gamma_{03} & \frac{2}{3} \left(1 + \frac{1}{\sqrt{2}}\right) \\ \gamma_{04} & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{12} \end{bmatrix} \quad (5.29)$$

Solving for the $\{\gamma_{0j}\}$ gives

$$\begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \\ \gamma_{04} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{6} \cdot \frac{1}{1 - \frac{1}{\sqrt{2}}} \\ \frac{2}{3} - \frac{1}{6} \cdot \frac{1}{1 - \frac{1}{\sqrt{2}}} \\ 0 \end{bmatrix} \quad (5.30)$$

In addition, we have the constraints

$$\begin{aligned} \beta_2 + \delta_2 + \xi_2 &= 1, & \beta &= \frac{1}{4}, & \beta_1 + \delta_1 &= \beta, \\ \beta_2 - 2\left(1 + \frac{1}{\sqrt{2}}\right)\delta_1 &= 0, & \text{and} & & & \\ \left(1 - \frac{1}{\sqrt{2}}\right)\beta + \left(1 + \frac{1}{\sqrt{2}}\right)\beta_1 + \frac{1}{2}\beta_2 &= \frac{1}{2}. \end{aligned} \quad (5.31)$$

If (5.31) is satisfied, $\varepsilon(hy_1' - hy_0') = O(h^6)$. Since only three entries to the function are desired for the case

$$f(x, y, y') = f(x, y),$$

set $\delta_1 = 0$, as in the Collatz procedure.

The constant δ_2 is chosen to be zero, arbitrarily.

Hence

$$\begin{aligned} \xi_2 &= 1, & \text{and} \\ \beta_2 &= 0. \end{aligned}$$

The procedure is now uniquely determined and is set forth in table (5.32).

x_i	y_i	$v_i = hy_i'$	$k_i = \frac{h^2}{2} f(x_i, y_i, y_i')$
x_0	y_0	v_0	k_1
$x_0 + \frac{1}{2}h$	$y_0 + \frac{v_0}{2} + \frac{k_1}{4}$	$v_0 + k_1$	k_2
$x_0 + \frac{1}{2}h$	$y_0 + \frac{v_0}{2} + \frac{k_1}{4}$	$v_0 + (-1 + \sqrt{2})k_1$ $+ (2 - \sqrt{2})k_2$	k_3
$x_0 + h$	$y_0 + v_0 + k_3$	$v_0 - \sqrt{2}k_2 +$ $(2 + \sqrt{2})k_3$	k_4
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k'$	$k = \frac{1}{3}k_1 + \frac{1}{3}\left(\frac{1}{2 - \sqrt{2}}\right)k_2$ $+ \frac{1}{3}\left(2 - \frac{1}{2 - \sqrt{2}}\right)k_3$ $k' = \frac{1}{3}\left(k_1 + 2\left(1 - \frac{1}{\sqrt{2}}\right)k_2\right.$ $\left. + 2\left(1 + \frac{1}{\sqrt{2}}\right)k_3 + k_4\right)$

(5.32)

Error components in this procedure are as follows:

$$\begin{aligned} \epsilon(y_1 - y_0) = & -\frac{1}{60}a - \frac{1}{60}b - \left(\frac{1}{3\sqrt{2}} - \frac{3}{20}\right)c \\ & - \frac{1}{30}d - \left(\frac{1}{6\sqrt{2}} - \frac{11}{120}\right)(e+f) - \frac{1}{60}g + O(h^6) \end{aligned} \quad (5.33)$$

and

$$\epsilon(hy_1' - hy_0') = (v_1 - v_0) = O(h^6) \quad (5.34)$$

For the case

$$f(x, y, y') = f(x, y),$$

then $k_2 = k_3$, and the procedure becomes identical to that of Collatz.

C. A Method Based on Boulton, Eqn. (3.26)

The constraint matrix equation is formulated, as before, using as a starting point the characteristics of the first-order Boulton scheme:

$$\begin{aligned} \gamma_{11} &= \frac{1}{4} ; \\ \gamma_{12} &= \frac{3}{4} ; \quad \alpha = \frac{1}{3} ; \quad \gamma = \frac{1}{3} ; \\ \gamma_{13} &= \frac{3}{4} ; \quad \alpha_1 = \frac{2}{3} ; \quad \gamma_1 = -\frac{1}{3} ; \quad \epsilon_1 = 1 ; \\ \gamma_{14} &= \frac{1}{4} ; \quad \alpha_2 = 1 \quad \gamma_2 = 1 ; \quad \epsilon_2 = 1 ; \quad \tau_2 = 1. \end{aligned} \quad (5.35)$$

Given the equation

$$\frac{d^2 y}{dx^2} = f(x, \frac{dy}{dx}) , \quad \text{this means that the}$$

solution is accomplished in the manner

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_0, y_0') \\ k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{3} h, y_0' + \frac{2}{3} \frac{k_1}{h}) \\ k_3 &= \frac{h^2}{2} f(x_0 + \frac{2}{3} h, y_0' - \frac{2}{3} \frac{k_1}{h} + 2 \frac{k_2}{2}) \\ k_4 &= \frac{h^2}{2} f(x_0 + h, y_0' + 2 \frac{k_1}{h} - 2 \frac{k_2}{h} + 2 \frac{k_3}{h}) \\ y_1' &= y_0' + \frac{1}{4} (\frac{k_1}{h} + 3 \frac{k_2}{h} + 3 \frac{k_3}{h} + 4 \frac{k_4}{h}) , \end{aligned}$$

in accordance with (4.53), which is identical to the first-order Boulton technique (3.26) with y' substituted for y .

However, the swing equation is of the form

$$\frac{d^2 y}{dx^2} = f(x, y, \frac{dy}{dx})$$

which necessitates, as in previous sections, the use of (4.51) to derive a suitable method of integration.

The constraint equations applied to (5.35) are:

$$\begin{bmatrix}
 1 & 1 & 1 & 1 \\
 0 & \frac{1}{3} & \frac{2}{3} & 1 \\
 0 & \frac{1}{9} & \frac{4}{9} & 1 \\
 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \xi_2 \\
 0 & 0 & \frac{2}{3} & \frac{2}{3} \\
 \dots & \dots & \dots & \dots \\
 1 & \frac{1}{27} & \frac{8}{27} & 1 \\
 0 & \frac{\beta}{3} & \frac{2}{3}(\beta_1 + \delta_1) & \beta_2 + \delta_2 + \xi_2 \\
 0 & 0 & \frac{4}{9} & \frac{2}{3} \\
 0 & 0 & \frac{\delta_1}{3} & \frac{\delta_2}{3} + \frac{2\xi_2}{3} \\
 0 & 0 & \frac{1}{9} & \frac{1}{3} \\
 0 & 0 & \beta & -\beta + (\beta_1 + \delta_1) \\
 0 & 0 & 0 & \frac{1}{3}
 \end{bmatrix} \cdot \begin{bmatrix} \gamma_{01} & \frac{1}{4} \\ \gamma_{02} & \frac{3}{4} \\ \gamma_{03} & \frac{3}{4} \\ \gamma_{04} & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \dots & \dots \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{12} \end{bmatrix} \quad (5.36)$$

From these equations we have

$$\begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \\ \gamma_{04} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \\ 0 \end{bmatrix} \quad (5.37)$$

and also

$$3\beta + 3(\beta_1 + \delta_1) + (\beta_2 + \delta_2 + \xi_2) = \frac{8}{3} \quad (5.38)$$

$$\beta + 2(\beta_1 + \delta_1) + (\beta_2 + \delta_2 + \xi_2) = 2 \quad (5.39)$$

$$3\delta_1 + \delta_2 + 2\xi_2 = 2 \quad (5.40)$$

Again, if we aim for three function evaluations per iteration in the case of a function in which the first derivative does not appear explicitly, it is advantageous to set

$$\begin{bmatrix} \beta_2 \\ \delta_2 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ \frac{1}{4} \end{bmatrix} \quad (5.41)$$

$$\text{From (5.38) - (5.40)} \quad \delta_1 = \frac{1}{3}, \quad \beta = \beta_1 = \frac{1}{9} \quad (5.42)$$

The equations represented by the lower sub-matrices in (5.36) then are:

$$\begin{bmatrix} 0 & \frac{1}{27} & \frac{8}{27} & 1 \\ 0 & \frac{1}{27} & \frac{8}{27} & 1 \\ 0 & 0 & \frac{4}{9} & \frac{2}{3} \\ 0 & 0 & \frac{1}{9} & \frac{1}{3} \\ 0 & 0 & \frac{1}{9} & \frac{1}{3} \\ 0 & 0 & \frac{1}{9} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{5}{54} & \frac{1}{2} \\ \frac{5}{54} & \frac{1}{2} \\ \frac{1}{9} & \frac{1}{2} \\ \frac{1}{36} & \frac{1}{6} \\ \frac{1}{36} & \frac{1}{6} \\ \frac{1}{36} & \frac{1}{6} \\ 0 & \frac{1}{12} \end{bmatrix} \quad (5.43)$$

Error of order h^5 is given by:

$$\begin{aligned} \epsilon(y_1 - y_0) = & -\frac{1}{135} a - \frac{1}{135} b + \frac{1}{90} c + \frac{1}{180} d - \frac{1}{180} e \\ & - \frac{1}{180} f - \frac{1}{60} g + O(h^6) \end{aligned} \quad (5.44)$$

and

$$\epsilon(hy'_1 - hy'_0) = (v_1 - v_0) = O(h^6) \quad (5.45)$$

The completed process is given by:

x_i	y_i	$v_i = hy'_i$	$k_i = \frac{h^2}{2} f(x_i, y_i, v_i/h)$	
x_0	y_0	v_0	k_1	
$x_0 + \frac{1}{3}h$	$y_0 + \frac{1}{3}v_0 + \frac{1}{9}k_1$	$v_0 + \frac{2}{3}k_1$	k_2	
$x_0 + \frac{2}{3}h$	$y_0 + \frac{2}{3}v_0 + \frac{1}{9}k_1 + \frac{1}{3}k_2$	$v_0 + \frac{2}{3}k_1 + 2k_2$	k_3	
$x_0 + h$	$y_0 + v_0 + k$	$v_0 + 2k_1 - 2k_2 + 2k_3$	k_4	
			$k = \frac{1}{4}(k_1 + 2k_2 + k_3)$	
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k'$	$k' = \frac{1}{4}(k_1 + 3k_2 + 3k_3 + k_4)$	(5.46)

Example 5.2

The three second-order methods are compared by a numerical integration of the swing equation of Example 3.1, assuming a fault clearing time of 0.60 second. Computer results for values of step length h from 0.20 sec. to 0.003125 sec. are given in Appendix (C). Figure 5.2 shows some of the resulting curves.

As predicted, the process of (5.25) gives poorer results in this application than do the other two methods.

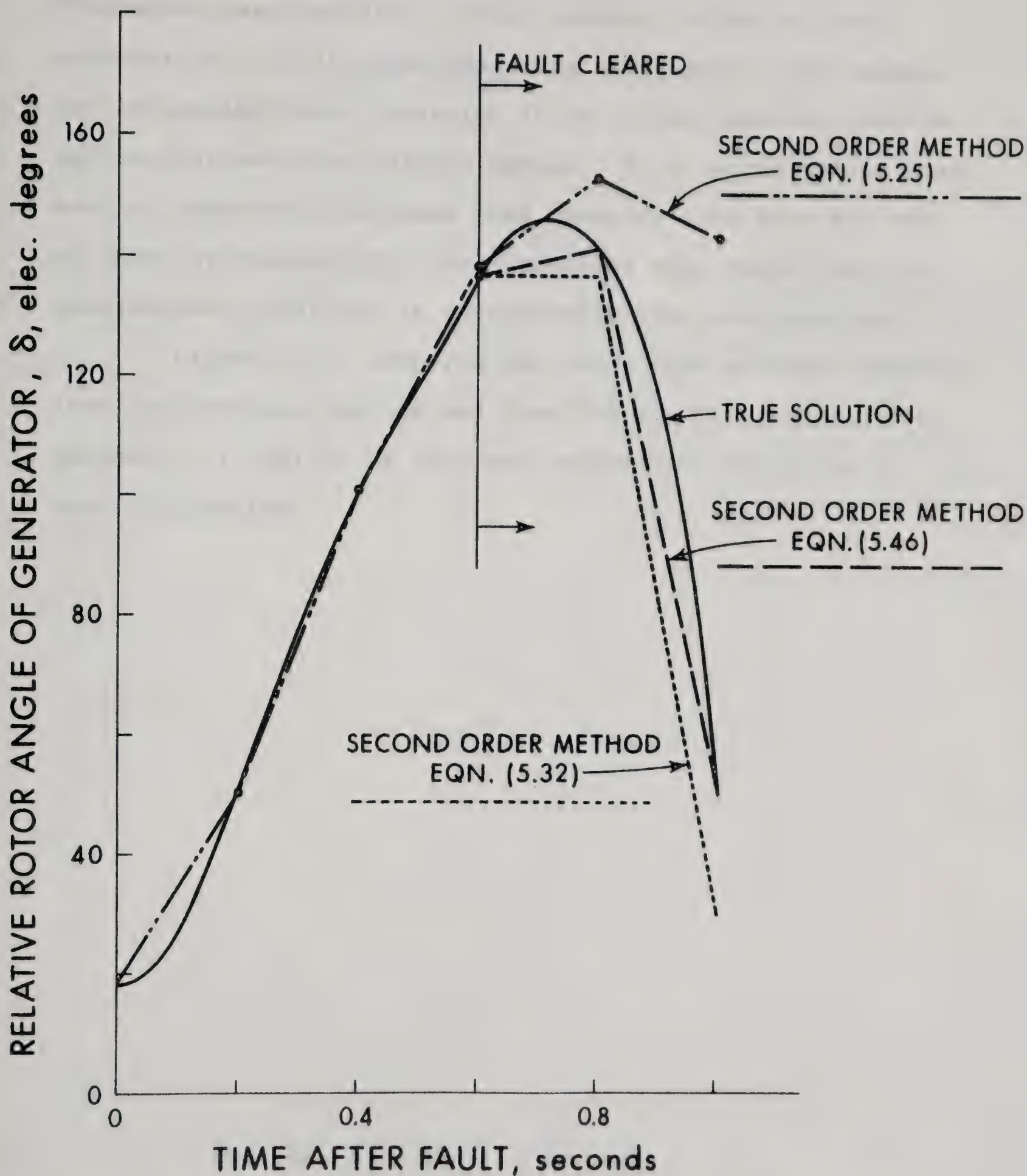


FIG. 5.2 SWING CURVES OF EXAMPLE 5.2 - CLEAR.
TIME 0.60 SEC

The method based on Gill, (5.32) reduces to the Collatz procedure of (5.17) since the swing equation in this example has no damping term. Equation (5.46) gives superior results - better than even the Collatz method. It is worth noting that each of these methods takes less computer time than did any of those of Chapter III. Each requires only three function entries per iteration, in comparison to the previous four.

Figure (5.3) compares the three most accurate methods from the previous section and from Chapter III, showing the process of (5.46) to be the most suitable of the three in this application.

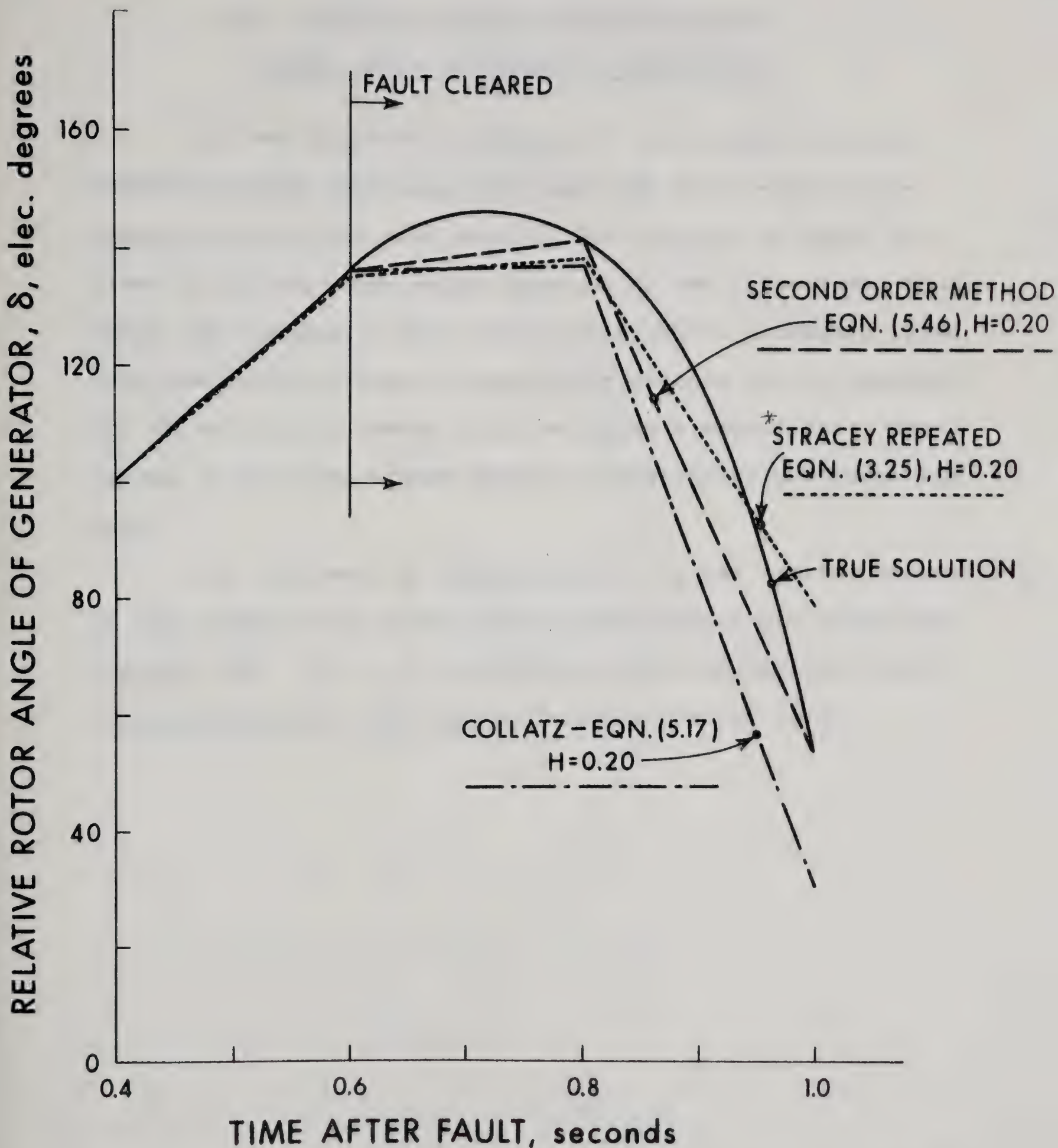


FIG. 5.3 SWING CURVES OF EXAMPLE 5.2 - CLEAR. TIME 0.60 SEC

VI. INTERVAL CHANGING PROCEDURES FOR
SECOND ORDER DIFFERENTIAL EQUATIONS

It was observed in Chapter V that Runge-Kutta procedures derived especially for use with second-order differential equations gave results far superior to those derived by reducing the swing equation to two first-order equations and applying a first-order Runge-Kutta procedure twice. This observation gives an optimistic picture of the possibility of solving the swing equation using a second-order procedure in the same manner Merson's first-order procedure was used.

To this end, an expression for k_5 was carried through in the second-order derivation of the Runge-Kutta algorithm (Chapter IV). If k_5 is included in the formulation of the constraint matrix equation, it becomes that of (6.1).

[illegible]

The operational table, defining the symbols, is as follows:

x_i	y_i	$v_i = hy_i'$	k_i
x_0	y_0	v_0	k_1
$x_0 + \alpha h$	$y_0 + \alpha v_0 + \beta k_1$	$v_0 + 2(\gamma k_1)$	k_2
$x_0 + \alpha_1 h$	$y_0 + \alpha_1 v_0 + \beta_1 k_1 + \delta_1 k_2$	$v_0 + 2(\gamma_1 k_1 + \epsilon_1 k_2)$	k_3
$x_0 + \alpha_2 h$	$y_0 + \alpha_2 v_0 + \beta_2 k_1 + \delta_2 k_2 + \xi_2 k_3$	$v_0 + 2(\gamma_2 k_1 + \epsilon_2 k_2 + \tau_2 k_3)$	k_4
$x_0 + \alpha_3 h$	$y_0 + \alpha_3 v_0 + \beta_3 k_1 + \delta_3 k_2 + \xi_3 k_3 + \lambda_3 k_4$	$v_0 + 2(\gamma_3 k_1 + \epsilon_3 k_2 + \tau_3 k_3 + \phi_3 k_4)$	k_5
			$k = \sum_{i=1}^5 \gamma_{0i} k_i$
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$	$v_1 = v_0 + k'$	$k' = \sum_{j=1}^5 \gamma_{1j} k_j$

(6.2)

In each case $k_i = \frac{h^2}{2} f(x_i, y_i, y_i')$.

Two interval-changing procedures will be developed. The first has, as a basis, the first-order Merson integration scheme; the second is based on the Boulton procedure.

A. A Process Based on Merson, Eqn. (3.31)

The first-order Merson procedure is given by the following steps.

$$\begin{aligned}
 k_1' &= hf(x_0, y_0') \\
 k_2' &= hf(x_0 + \frac{1}{3}h, y_0' + \frac{1}{3}k_1')
 \end{aligned}
 \tag{6.3}$$

$$k_3' = hf(x_0 + \frac{1}{3}h, y_0' + \frac{1}{6}k_1' + \frac{1}{6}k_2')$$

$$k_4' = hf(x_0 + \frac{1}{2}h, y_0' + \frac{1}{8}k_1' + \frac{3}{8}k_3')$$

$$k_5' = hf(x_0 + h, y_0' + \frac{1}{2}k_1' - \frac{3}{2}k_3' + 2k_4')$$

and

$$y_1' = y_0' + \frac{1}{6}(k_1' + 4k_4' + k_5') + O(h^5)$$

$$30\epsilon(y_1' - y_0') \approx 2k_1' - 9k_3' + 8k_4' - k_5' \quad (6.4)$$

The above form was easily obtained from (3.31) by assuming the first-order equation

$$\frac{dy'}{dx} = f(x, y').$$

If we multiply both equations of (6.4) by $\frac{h}{2}$ and let $k_i = \frac{h}{2} k_i'$, the $\{k_i\}$ are evaluated in the manner

$$\begin{aligned} k_1 &= \frac{h^2}{2} f(x_0, y_0') \\ k_2 &= \frac{h^2}{2} f(x_0 + \frac{1}{3}h, y_0' + \frac{2}{3}\frac{k_1}{h}) \\ k_3 &= \frac{h^2}{2} f(x_0 + \frac{1}{3}h, y_0' + \frac{1}{3}\frac{k_1}{h} + \frac{1}{3}\frac{k_2}{h}) \\ k_4 &= \frac{h^2}{2} f(x_0 + \frac{1}{2}h, y_0' + \frac{1}{4}\frac{k_1}{h} + \frac{3}{4}\frac{k_3}{h}) \\ k_5 &= \frac{h^2}{2} f(x_0 + h, y_0' + \frac{k_1}{h} - \frac{3k_3}{h} + \frac{4k_4}{h}) \end{aligned} \quad (6.5)$$

and the integration is accomplished in the manner

$$hy_1' = hy_0' + \frac{1}{3}(k_1 + 4k_4 + k_5) + O(h^5); \quad \text{and} \quad (6.6)$$

$$15\epsilon(hy_1' - hy_0') \approx 2k_1 - 9k_3 + 8k_4 - k_5. \quad (6.7)$$

The equation we must solve is

$$\frac{d^2 y}{dx^2} = f(x, y, y') , \quad (6.8)$$

so that the expressions of (6.5) must be altered somewhat to allow for the express appearance of y in the function.

From (6.5), referring to table (6.2), we have

$$\begin{aligned} \gamma_{11} &= \frac{1}{3} ; \\ \gamma_{12} &= 0 ; \quad \gamma = \frac{1}{3} ; \quad \alpha = \frac{1}{3} ; \\ \gamma_{13} &= 0 ; \quad \gamma_1 = \frac{1}{6} ; \quad \epsilon_1 = \frac{1}{6} ; \quad \alpha_1 = \frac{1}{3} ; \\ \gamma_{14} &= \frac{4}{3} ; \quad \gamma_2 = \frac{1}{8} ; \quad \epsilon_2 = 0 ; \quad \tau_2 = \frac{3}{8} ; \quad \alpha_2 = \frac{1}{2} ; \\ \gamma_{15} &= \frac{1}{3} ; \quad \gamma_3 = \frac{1}{2} ; \quad \epsilon_3 = 0 ; \quad \tau_3 = -\frac{3}{2} \quad \tau_4 = 2 ; \quad \alpha_3 = 1 . \end{aligned} \quad (6.9)$$

Substitution of these values into the constraint matrix equation (6.1) gives equations (6.10).

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{4} & 1 \\ 0 & \beta & \beta_1 + \delta_1 & \beta_2 + \delta_2 + \xi_2 & \beta_3 + \delta_3 + \xi_3 + \lambda_3 \\ 0 & 0 & \frac{1}{9} & \frac{1}{4} & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \frac{1}{27} & \frac{1}{27} & \frac{1}{8} & 1 \\ 0 & \frac{\beta}{3} & \frac{\beta_1 + \delta_1}{3} & \frac{1}{2}(\beta_2 + \delta_2 + \xi_2) & \beta_3 + \delta_3 + \xi_3 + \lambda_3 \\ 0 & 0 & \frac{1}{27} & \frac{1}{8} & 1 \\ 0 & 0 & \frac{\delta_1}{3} & \frac{1}{2}(\delta_2 + \xi_2) & \frac{1}{3}(\xi_3 + \delta_3) + \frac{\lambda_3}{2} \\ 0 & 0 & \frac{1}{54} & \frac{1}{24} & \frac{1}{3} \\ 0 & 0 & \frac{\beta}{6} & 3(\beta_1 + \delta_1)/8 & -\frac{3}{2}(\beta_1 + \delta_1) + 2(\beta_2 + \delta_2 + \xi_2) \\ 0 & 0 & 0 & \frac{1}{48} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \\ \gamma_{04} \\ \gamma_{05} \end{bmatrix} \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ \frac{4}{3} \\ \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ \frac{1}{3} & 1 \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \\ \dots & \dots \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{2} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{6} \\ - & \frac{1}{12} \end{bmatrix}$$

From these equations, the following relationships are determined -

$$\begin{bmatrix} \gamma_{01} \\ \gamma_{02} \\ \gamma_{03} \\ \gamma_{04} \\ \gamma_{05} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \\ \frac{2}{3} \\ 0 \end{bmatrix} ; \quad \begin{aligned} \frac{2}{3}(\beta_2 + \delta_2 + \xi_2) &= \frac{1}{6} ; \\ \beta_3 + \delta_3 + \xi_3 + \lambda_3 &= 1 ; \\ \frac{2}{3}(\delta_2 + \xi_2) + \frac{1}{9}(\xi_2 + \delta_3) + \frac{1}{6}\lambda_3 &= 0 . \end{aligned} \quad (6.11)$$

Again, the swing equation with no damping term will be considered. Then

$$f_u = \frac{\partial f}{\partial y'} = 0 ,$$

and the terms involving f_u may be neglected. In order to save one function entry per integration step, it is expedient to place the constraint

$$\beta = \beta_1 + \delta_1 ; \quad \delta_1 = 0 ,$$

on the system of equations. Let us define μ_i such that the iteration error in y , $\epsilon(y_1 - y_0)$ is given by

$$\begin{aligned} \epsilon(y_1 - y_0) &= \sum_{j=1}^5 \mu_j k_j + O(h^6) \\ &= \epsilon_1 a + \epsilon_2 b + \epsilon_3 d + O(h^6) . \end{aligned} \quad (6.12)$$

Error components a , b , and d are identical to those used in the preceding two chapters.

Our system of equations then is in

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{2} & 1 \\ 0 & \frac{1}{9} & \frac{1}{9} & \frac{1}{4} & 1 \\ 0 & \beta & \beta & \frac{1}{4} & 1 \\ 0 & \frac{1}{27} & \frac{1}{27} & \frac{1}{8} & 1 \\ 0 & \frac{\beta}{3} & \frac{\beta}{3} & \frac{1}{8} & 1 \\ 0 & 0 & 0 & \frac{1}{2}(\delta_2 + \xi_2) & \frac{1}{3}(\xi_3 + \delta_3) + \frac{1}{2}\lambda_3 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} & \mu_1 & \frac{1}{3} \\ 0 & \mu_2 & 0 \\ 0 & \mu_3 & 0 \\ \frac{2}{3} & \mu_4 & \frac{4}{3} \\ 0 & \mu_5 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ \frac{1}{3} & 0 & 1 \\ \frac{1}{6} & 0 & \frac{2}{3} \\ \frac{1}{6} & 0 & \frac{2}{3} \\ \frac{1}{10} + \epsilon_1 & \epsilon_1 & \frac{1}{2} \\ \frac{1}{10} + \epsilon_2 & \epsilon_2 & \frac{1}{2} \\ \frac{1}{30} + \epsilon_3 & \epsilon_3 & \frac{1}{6} \end{bmatrix} \quad (6.13)$$

Solving gives

$$\begin{aligned} \delta_2 + \xi_2 &= \frac{3}{8} \\ \beta_2 &= -\frac{1}{8} \\ \beta &= \frac{1}{9} \end{aligned} \quad (6.14)$$

$$\begin{aligned} \epsilon_1 &= -\frac{1}{60} \\ \epsilon_2 &= -\frac{1}{60} \\ \epsilon_3 &= +\frac{1}{20} \end{aligned} \quad \underline{\mu} = \frac{1}{20} \begin{bmatrix} 2 \\ -9 \\ 0 \\ +8 \\ -1 \end{bmatrix} \quad \text{or} \quad \frac{1}{20} \begin{bmatrix} 2 \\ 0 \\ -9 \\ 8 \\ -1 \end{bmatrix} \quad (6.15)$$

In the case under consideration $k_2 = k_3$ so that $\mu_2 + \mu_3 = -9$ is the only constraint on μ_2 and μ_3 . The coefficient β_3 was chosen to be zero, giving

$$\begin{aligned} \delta_3 + \xi_3 &= 3, \\ \lambda_3 &= -2. \end{aligned} \quad (6.16)$$

The operational table, deleting k_3 and setting $k_3' = k_4$, $k_4' = k_5$, $k_1' = k_1$ and $k_2' = k_2 = k_3$, becomes with $v_i = hy_i'$:

x_i	y_i	$k_i' = \frac{h^2}{2} f(x_i, y_i)$	(6.17)
x_0	y_0	k_1'	
$x_0 + \frac{1}{3}h$	$y_0 + \frac{1}{3}v_0 + \frac{k_1'}{9}$	k_2'	
$x_0 + \frac{1}{2}h$	$y_0 + \frac{1}{2}v_0 - \frac{1}{8}k_1' + \frac{3}{8}k_2'$	k_3'	
$x_0 + h$	$y_0 + v_0 + 3k_2' - 2k_3'$	k_4'	
$x_1 = x_0 + h$	$y_1 = y_0 + v_0 + k$ $v_1 = v_0 + k'$	$k_e = 2k_1' - 9k_2' + 8k_3' - k_4'$ $k = \frac{1}{3}(k_1' + 2k_3')$ $k' = \frac{1}{3}(k_1' + 4k_3' + k_4')$ $\epsilon(y_1 - y_0) = \frac{1}{20}k_e + O(h^6)$	

B. A Process Based on Eqn. (5.46)

Using a somewhat similar technique to that of Chapter VI (A), the following operational table was derived:

x_i	y_i	$v_i = hy_i'$	k_i
x_0	y_0	v_0	k_1
$x_0 + \frac{1}{3}h$	$y_0 + \frac{v_0}{3} + \frac{1}{9}k_1$	$v_0 + \frac{2}{3}k_1$	k_2
$x_0 + \frac{2}{3}h$	$y_0 + \frac{v_0}{3} - \frac{1}{18}k_1 + \frac{1}{6}k_2$	$v_0 - \frac{2}{3}k_1 + \frac{4}{3}k_2$	k_3
$x_0 + \frac{2}{3}h$	$y_0 + \frac{2v_0}{3} + \frac{1}{9}k_1 + \frac{1}{3}k_2$	$v_0 - \frac{2}{3}k_1 + 2k_2$	k_4
$x_0 + h$	$y_0 + v_0 + \frac{1}{4}k_1 + \frac{1}{2}k_2$ $+ \frac{1}{4}k_4$	$v_0 + 2k_1 - 2k_2$ $+ 2k_4$	k_5
$x_1 = x_0 + h$	$y_1 - y_0 = v_0 + k$ $\epsilon(y_1 - y_0) = \frac{1}{10}k_e$ $+ O(h^6)$	$v_1 = v_0 + k'$	$k = \frac{1}{4}k_1 + \frac{1}{2}k_2 + \frac{1}{4}k_4$ $k' = \frac{1}{4}k_1 + \frac{3}{4}(k_2 + k_4)$ $+ \frac{1}{4}k_5$ $k_e = \frac{1}{3}k_1 - k_3 + k_4$ $- \frac{1}{3}k_5$

In each case $k_i = \frac{h^2}{2} f(x_i, y_i, y_i')$. (6.18)

It should be noted that the integration procedure is identical with that set forth in (5.46), which proved the most successful of the methods tested. The method is more versatile than (6.17) because it is applicable to a function containing the first derivative explicitly. In the latter case, however, five function entries are required per iteration, since

$$k_5 \neq \frac{h^2}{2} f(x_0 + h, y_1).$$

Example 6.1

Compare the interval-changing procedures of Eqns. (6.17) and (6.18) with respect to their speed and accuracy by integration of the swing equation of example 3.1, assuming a fault clearing time of 0.60 seconds.

The criterion used to evaluate the necessity of an interval change during the integration was identical to that set forth by Merson, as in Eqn. (3.32a). Figure 6.1 shows the procedure derived from (5.46) to be more suitable in this application. It requires six less function evaluations than does the other, yet achieves a more correct result at time one second after fault.

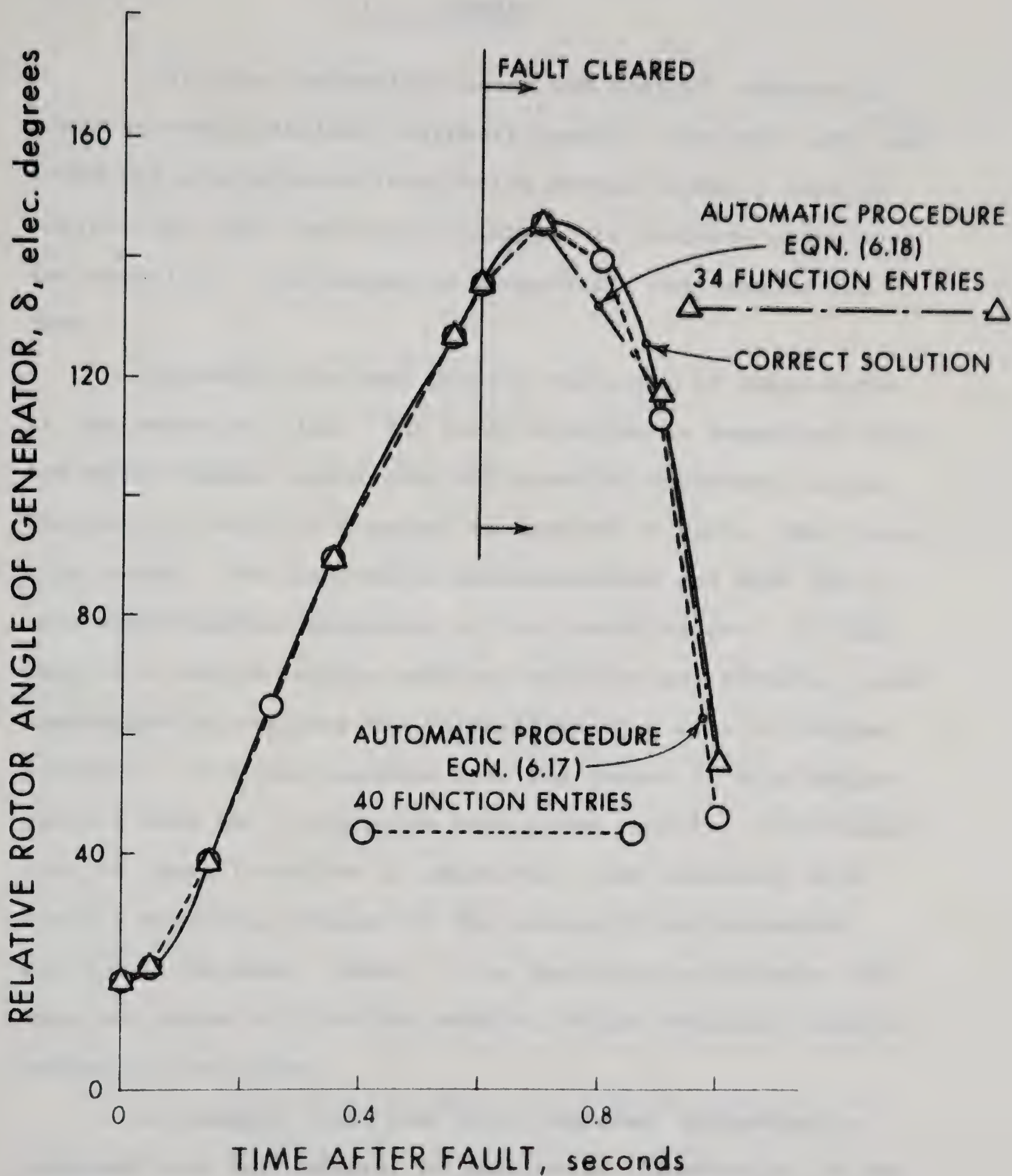


FIG. 6.1 SWING CURVES OF EXAMPLE 6.1 - CLEAR. TIME 0.60 SEC

VII. SUMMARY

With the increasing use of the digital computer in power systems planning, engineers have for the most part abandoned the step-by-step integration method formerly used in conjunction with the network analyzer to evaluate power system stability. The method of Runge-Kutta has come to the fore.

Presently the most popular variation of Runge-Kutta is the method of Gill. The swing equation is separated into two simultaneous first-order differential equations, as in Chapter II, and Gill's method is applied to each. When using this method, four successive approximations are made for δ ; each approximation dependent on the preceding one. In the case of a single machine problem, which arises commonly, each approximation requires the calculation of a sine or cosine function. In a multimachine case the number of sine/cosine entries made per integration step rises rapidly. The evaluation of these functions is relatively time consuming on a digital computer, because of the nature of the procedure which must be used - hence, it is important to minimize the required number of function entries, while retaining desired accuracy of solution.

In Example (7.1) the "Gill Repeated" procedure is compared with four others, to provide an illustration of the rapidity of each. The method of De Vogelaere, introduced in this example, is set forth in Appendix A.

Example 7.1

Using the equation of Example 3.1 with fault clearing time 0.60 second, perform as many integration steps as necessary to achieve an accuracy of 1.0 el. degrees in δ at time 1.0 second after fault. Compare the computational time for each of the following:

- i) Gill Repeated procedure - Eqn. (3.23)
- ii) Johnson Ward procedure - Eqn. (5.12)
- iii) Collatz procedure - Eqn. (5.17)
- iv) Second-order procedure - Eqn. (5.46)
- v) De Vogelaere procedure - Eqn. (A.1-.5)

The problem was approached by first calculating a swing curve with an interval size $h = 0.20$ second. Another curve was then calculated, using the same data but with step size 0.10 second. The values of delta at time one second after fault ($\delta_{1.0}$) on each curve were then compared - if these values differed by 1.0 el. degree or less, the calculation was stopped and the curves were printed out with a statement indicating the procedure had converged to the prescribed accuracy. If the values of $\delta_{1.0}$ differed by more than this amount, the step size was again halved, a new curve was calculated, and the test again applied.

The program was written in Fortran IV language and executed on an IBM 7040 digital computer. For some of the methods it was found that total calculation time was less

than one second. The entire problem was thus solved 100 times for each method, and the calculation time divided by 100. To give a more accurate comparison between each, Table 7.1 sets forth the results of this study, and computer output is shown in Appendix C.

Table 7.1
Results of Example 7.1

Method of Equation	3.23	5.12	5.17	5.46	A.1-A.5
Acceptable value of $\delta_{1.0}$, el. degrees	53.498	54.956	54.073	53.011	53.995
No. function entries necessary	84	164	63	20	85
Largest error value encountered	0.755	0.583	0.207	0.973	0.256
Necessary step size	0.05	0.025	0.05	0.2	0.025
7040 execution time in seconds	0.68	1.23	0.52	0.17	0.78

The method proposed by the writer takes less than fourteen percent of the time required by the method of Johnson and Ward and about twenty-five percent of the time required by the "Gill Repeated procedure" to solve the problem to the accuracy prescribed. Computational time increases rapidly as the number of machines involved in a stability study increases, so the time saving becomes considerable indeed for multimachine systems.

Example 7.2

For the system shown below, the swing equation is

$$\frac{1}{12,240} \frac{d^2 \delta}{dt^2} = 0.20 - 1.20 \sin \delta \quad \text{per unit.}$$

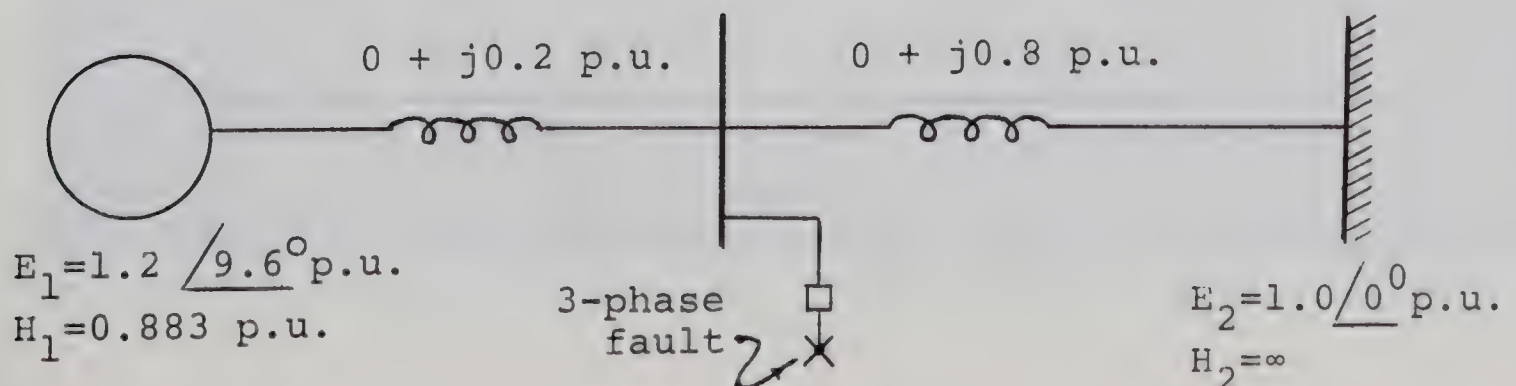
With the fault applied, this becomes

$$\frac{1}{12,240} \frac{d^2 \delta}{dt^2} = 0.200 \quad \text{per unit}$$

In this case, a three-phase bus fault is applied on bus A at $t = 0$ and removed at $t = 0.30$ second. Using each of the methods considered in Example 7.1, calculate and plot swing curves up to $t = 1.00$ second using time increments of 0.10, 0.05 and 0.025 seconds.

The computer output was written directly onto tape and an automatic plotting subroutine was used to give the curves shown in Fig. 7.1 - Fig. 7.5.

From the figures, it may be seen that the second-order method of (5.46) converges much more rapidly than do the others. The larger time increment again lessens the total computational time.



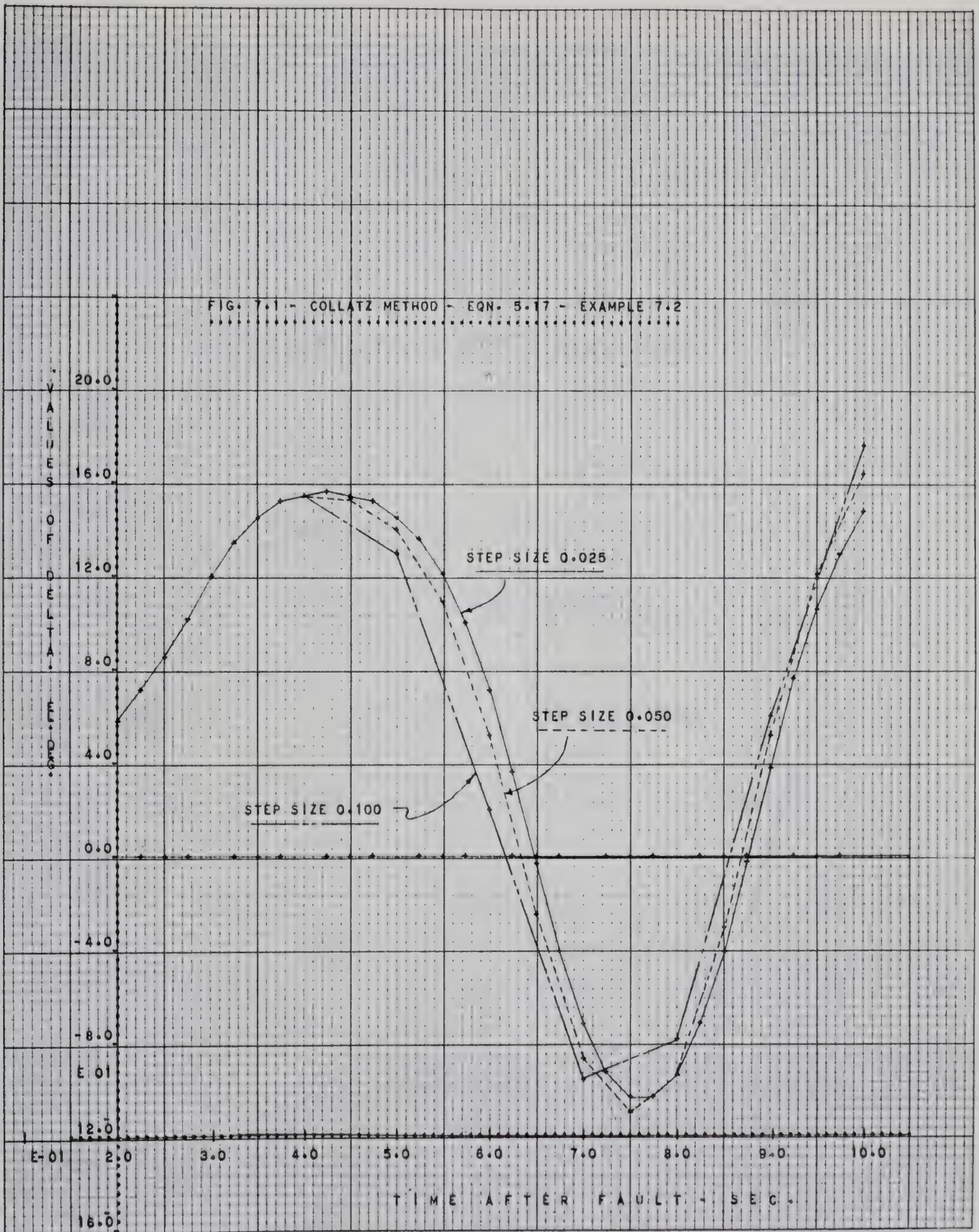


FIG. 7.2 - SECOND ORDER METHOD - EON = 5.46 - EXAMPLE 7.2

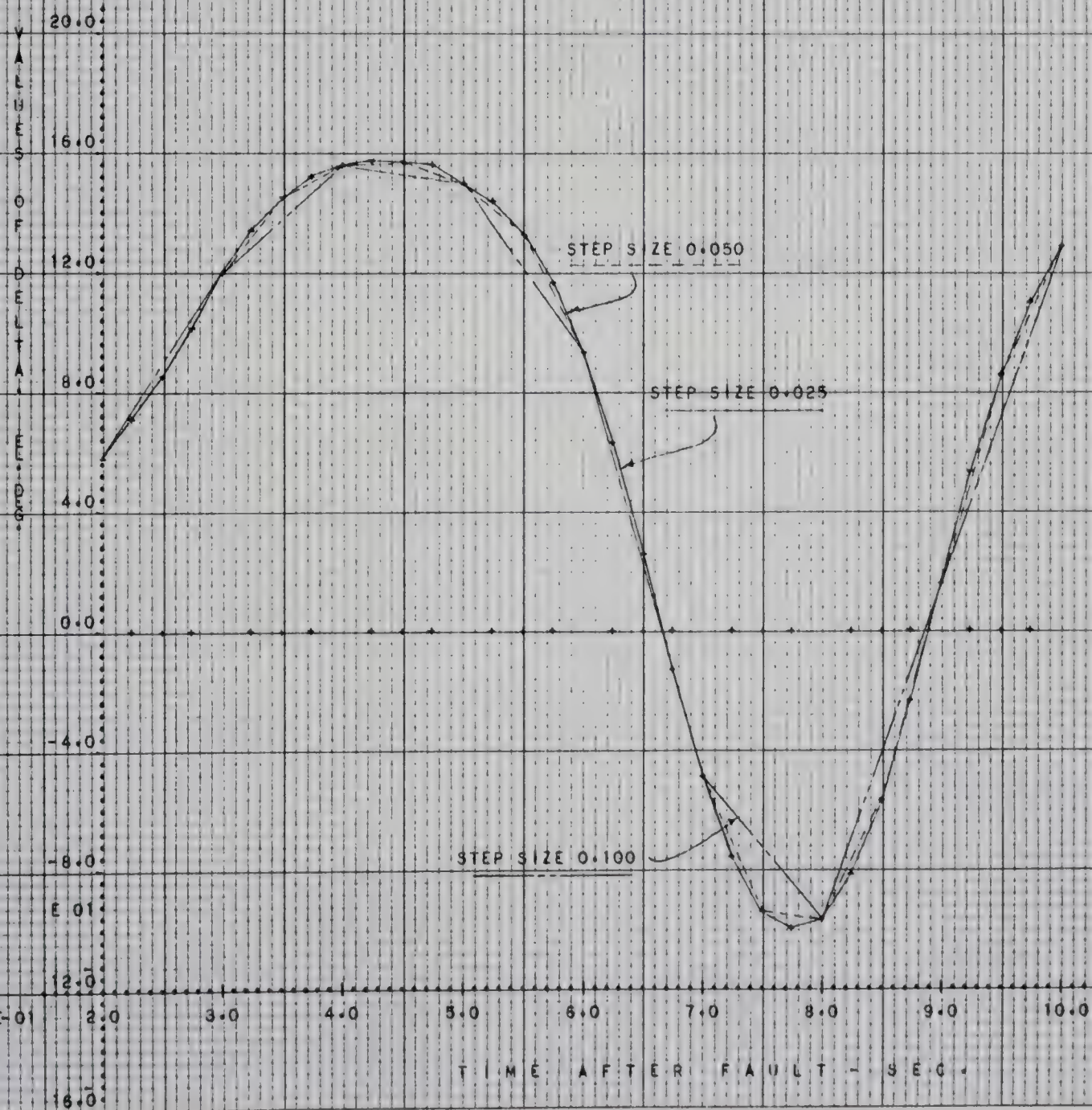


FIG. 7.3 - GILL REPEATED - EQN. 3.23 - EXAMPLE 7.2

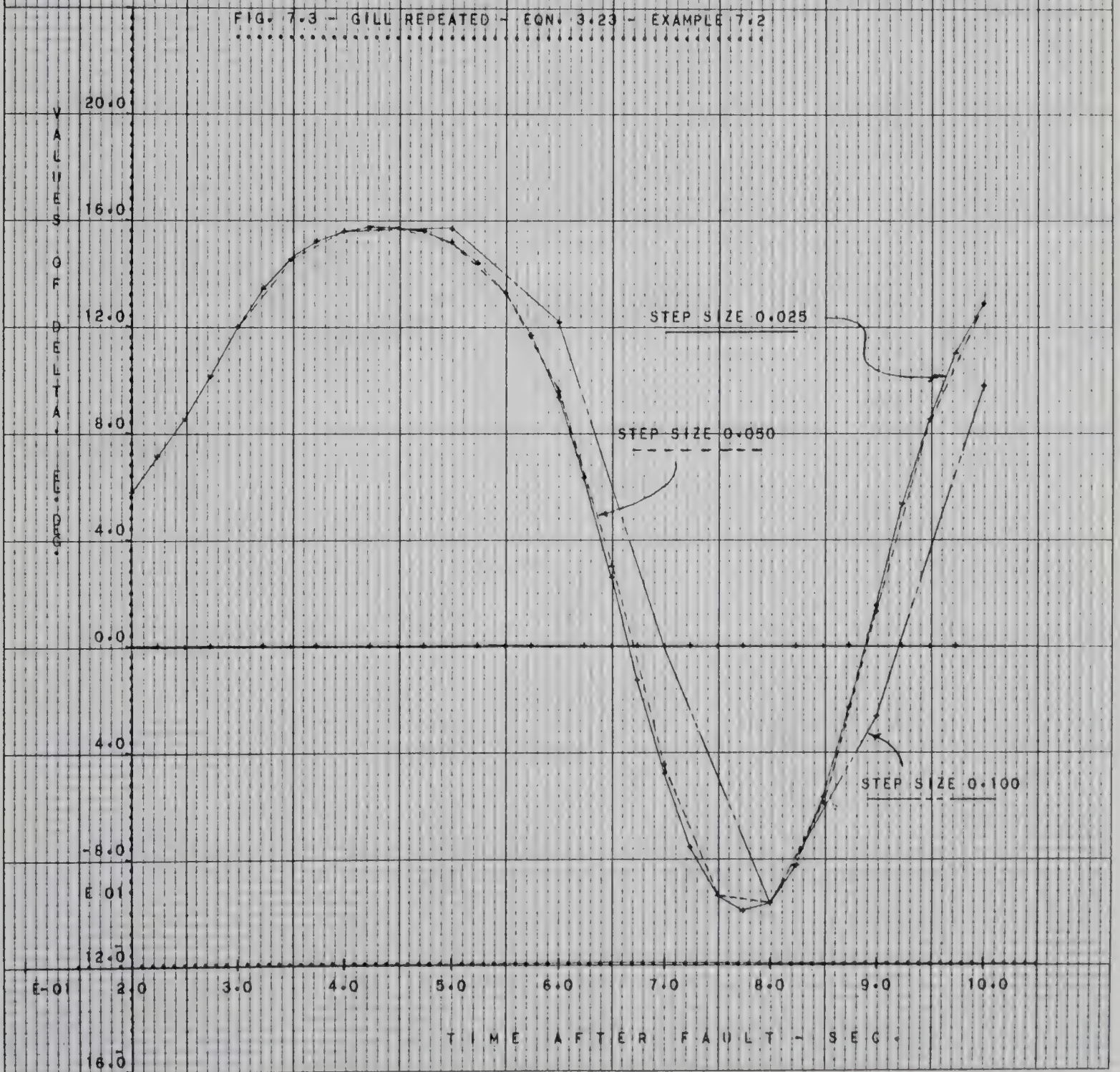


FIG. 7.4 - DE VOGELAERE - EON. A.1-A.5 - EXAMPLE 7.2

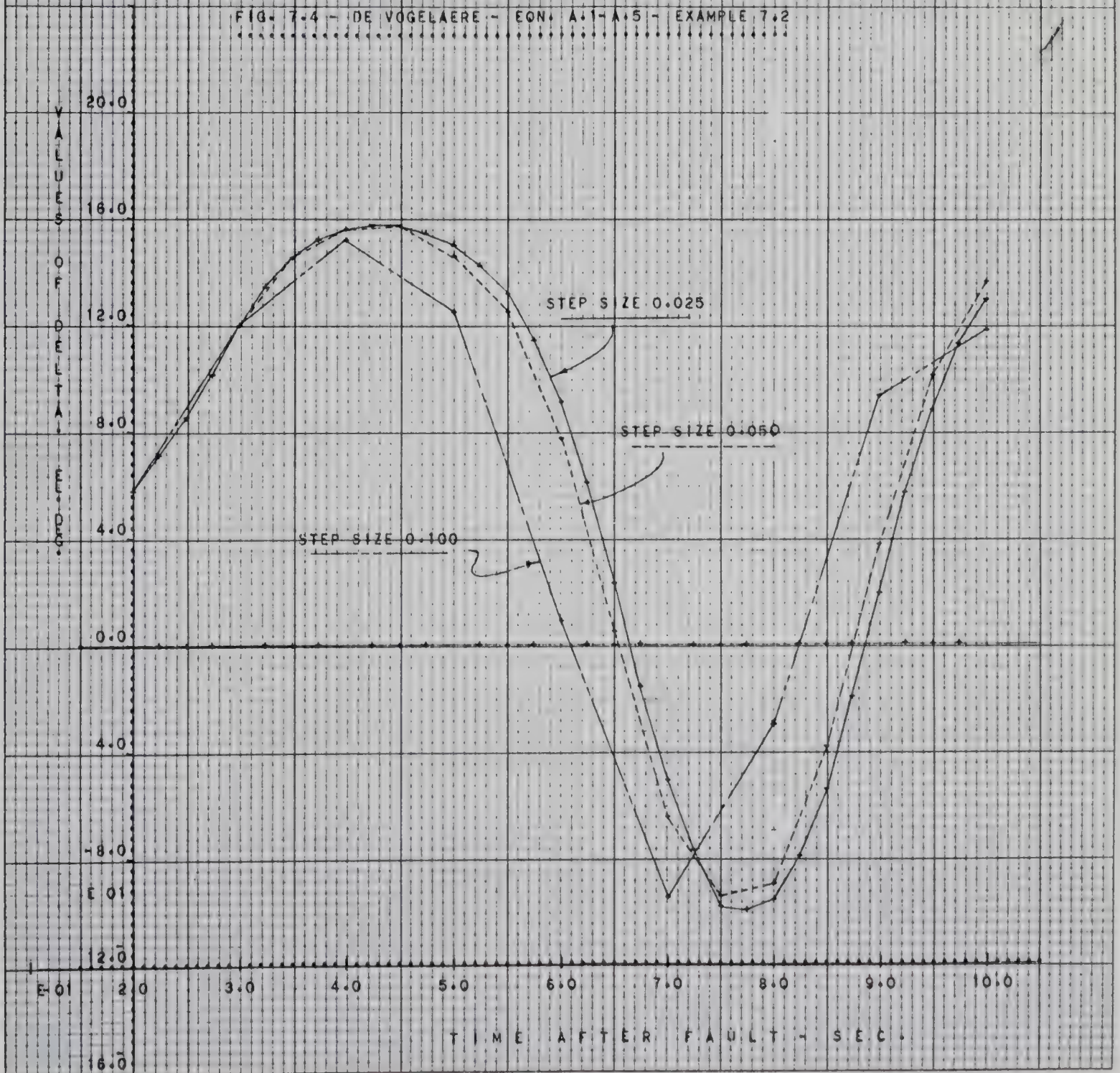
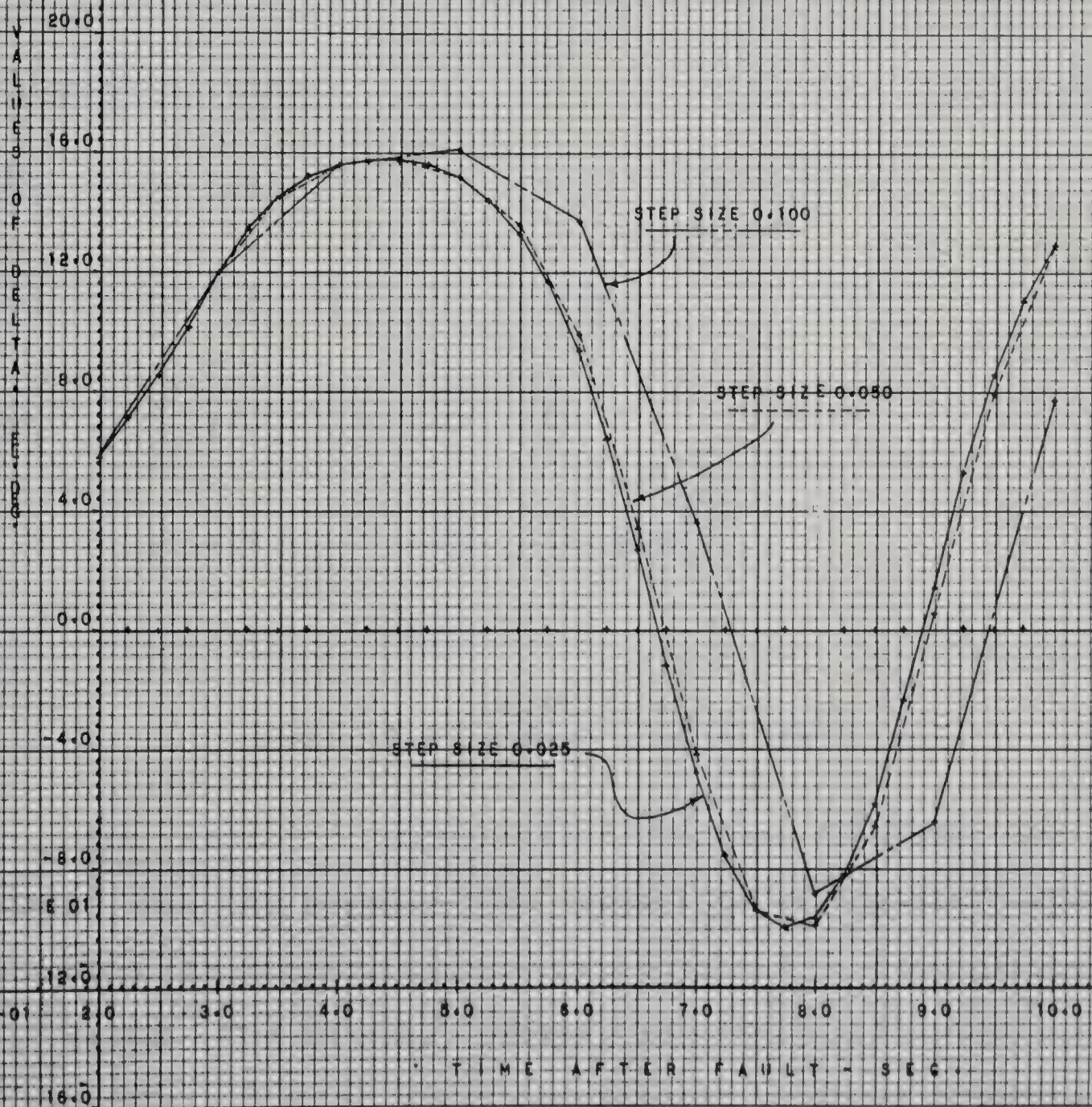


FIG. 7.5 - JOHNSON-WARD - EQN. 3.12 - EXAMPLE 7.2



Many Runge-Kutta procedures have been presented. Most give both increased efficiency and accuracy of integration over methods currently employed. Notable among the improved methods are Merson's variation, De Vogelaere's variation (Appendix A), the Collatz method, and the second-order technique of (5.46). In the examples studied, the method of (5.46) is superior. Restated in a step-by-step form to be applied to the equation

$$\frac{d^2\delta}{dt^2} = f(\delta),$$

this is to calculate

$$k_1 = \frac{h^2}{2} f(\delta)$$

$$k_2 = \frac{h^2}{2} f\left(\delta + \frac{v}{3} + \frac{k_1}{9}\right)$$

$$k_3 = \frac{h^2}{2} f\left(\delta + \frac{2v}{3} + \frac{k_1}{9} + \frac{k_2}{3}\right)$$

$$\delta = \delta_2 + v + \frac{k_1}{4} + \frac{k_2}{2} + \frac{k_3}{4} + O(h^5)$$

$$k_4 = \frac{h^2}{2} f(\delta), \quad \text{and}$$

$$v = h\omega = v + \frac{k_1}{4} + 3 \frac{k_2 + k_3}{4} + \frac{k_4}{4} + O(h^6).$$

The nature of Runge-Kutta techniques precludes any guarantee that the seventy-five percent to eighty-six percent time saving observed using this method could be accomplished in all cases. The scheme, however, is likely to give considerable benefit in most applications similar to those presented.

APPENDIX A

APPLICATION OF DE VOGELAERE'S METHOD TO THE
SOLUTION OF POWER SYSTEM STABILITY PROBLEMS

A numerical integration method⁽¹⁴⁾ devised by R. De Vogelaere in 1955 has achieved much popularity for solution of second-order differential equations such as

$$\frac{d^2 y}{dx^2} = f(x, y),$$

in which the first derivative does not appear explicitly.

Three relationships are required to describe the basic process-

$$y_{1/2} = y_0 + \frac{1}{2} h y_0' + \frac{1}{24} h^2 (4f_0 - f_{-1/2}) + O(h^4), \quad (A.1)$$

$$y_1 = y_0 + h y_0' + \frac{1}{6} h^2 (f_0 + 2f_{1/2}) + O(h^5), \text{ and} \quad (A.2)$$

$$y_1' = y_0' + \frac{1}{6} h (f_0 + 4f_{1/2} + f_1) + O(h^5). \quad (A.3)$$

For the initial integration step, note that

$$f_{-1/2} = f(x_{-1/2}, y_{-1/2}) = f(\overset{(x_0 - \frac{1}{2}h, y(x_0 - \frac{1}{2}h))}{\cancel{x_0 - \frac{1}{2}h, y(x_0 - \frac{1}{2}h)}}) \quad (A.4)$$

is required. A formula suggested by De Vogelaere which is sufficiently accurate for an estimate of this quantity is

$$y_{-1/2} = y_0 - \frac{1}{2} h y_0' + \frac{1}{8} h^2 f_0 + O(h^3) \quad (A.5)$$

The procedure is well-suited for digital computer application. Only two function evaluations are needed per integration step, and the step size may be changed with no extra steps required. A flow graph illustrating the technique is shown in Fig. A.1.

Note that in the calculation of k_2 , the value of k_2 from the previous interval is used. This is a drawback of the De Vogelaere procedure that Runge-Kutta techniques do not have - the fact that the iteration is not self-starting.

When comparing this method with others with respect to accuracy and time consumption, the fact that only two function entries are required here, per iteration, must be considered. As a standard of comparison, the total number of function entries required for integration of an equation over a given range is used. Table A.1 compares De Vogelaere's

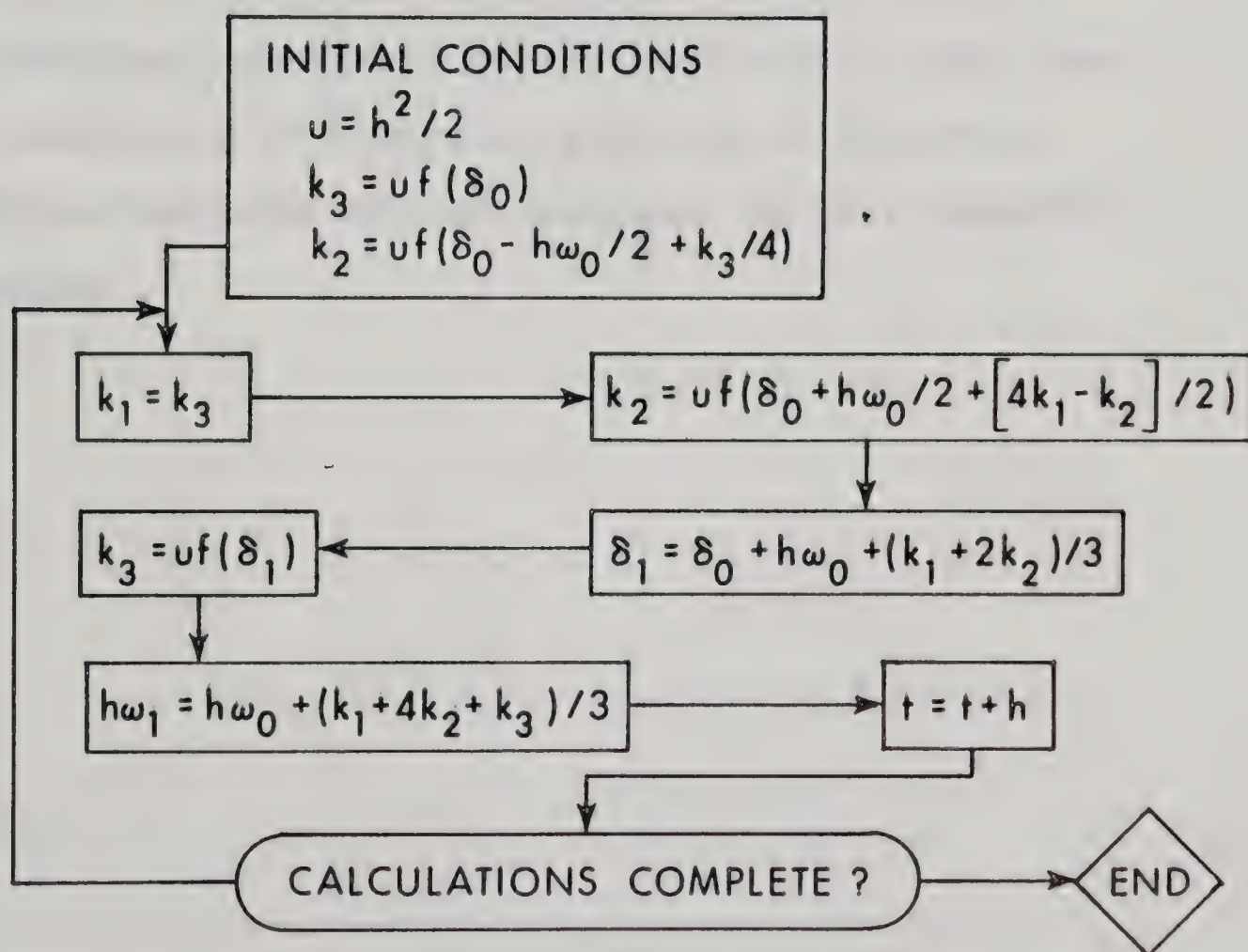


FIG. A.1 DeVogelaere's procedure to solve $d^2\delta/dt^2 = f(\delta)$

method, the presently-used Gill repeated procedure, and the second-order method developed by the writer, Eqn. (5.46). The problem solved is that of example 3.1, with a fault clearing time of 0.60 second.

The Gill Repeated procedure clearly requires more time than the others to do a less accurate job. The De Vogelaere procedure is a definite improvement over Gill's, but is not equal to the second-order method of (5.46) in this application. (5.46) is self-starting, which De Vogelaere's is not, and (5.46) has provision for solution of the swing equation with a damping term included; De Vogelaere's has not. The method is a definite improvement over those used to date, however, and no discussion of numerical solution of the swing equation would be complete without reference to this recently devised technique.

Table A.1
Values of δ (in electrical degrees) for Various Iteration Methods

Time, sec.	De Vogelaere		Gill Repeated(3.23)		Eqn. (5.46)		True
0.0	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.1		27.569	27.582		27.591	27.594	27.594
0.2	49.463	50.856	50.819	50.784	50.920	50.931	50.931
0.3		77.442	77.348		77.528	77.538	77.538
0.4	98.010	100.212	100.067	100.316	100.302	100.307	100.307
0.5		118.579	118.360		118.687	118.688	118.688
0.6	132.020	136.317	135.959	136.590	136.485	136.486	136.486
0.7		145.550	145.187		146.459	146.476	146.476
0.8	107.157	138.606	138.820	141.798	141.892	141.930	141.930
0.9		108.221	109.404		117.397	117.497	117.497
1.0	-37.090	36.058	38.404	53.011	53.984	54.283	54.283
Step size h	0.20	0.10	0.10	0.20	0.10		
No. Function entries req'd.	13	23	40	17	32		

APPENDIX BMETHOD OF APPLICATION OF RUNGE-KUTTA METHODSTO A MULTIMACHINE SYSTEM

The simplest form of the swing equation arises from consideration of one finite machine, connected through a transmission line to an infinite bus. As observed in Example 3.1, this leads to an equation

$$M \frac{d^2 \delta}{dt^2} = P_i - P_m \sin \delta = f(\delta) \quad (B.1)$$

Derivation of this equation assumes no damping and constant internal voltage as well as constant input. If damping and field variation are taken into account, there results the equation

$$M \frac{d^2 \delta}{dt^2} = f(t, \delta, \frac{d\delta}{dt}) \quad (B.2)$$

For a multimachine system, the power equation for the k^{th} machine, derived using the assumptions of (7.1), is

$$\begin{aligned} M_k \frac{d^2 \delta_k}{dt^2} &= P_i \Big|_k - \sum_{j=1}^n E_j E_k y_{kj} \cos (\theta_{kj} - \delta_k + \delta_j) \\ &= f(\delta_1, \delta_2, \dots, \delta_k, \dots, \delta_n) \end{aligned} \quad (B.3)$$

Here E_i is the voltage behind transient reactance of the i^{th} machine;

y_{kj}/θ_{kj} is the self- or transfer-admittance element of the network facing the machine, and

n is the total number of machines on the system.

The general equation for the k^{th} machine, taking into account both damping and field decrement, is of the form

$$M_k \frac{d^2 \delta_k}{dt^2} = f(t, \delta_1, \dots, \delta_k, \dots, \delta_n, \omega_1, \dots, \omega_k, \dots, \omega_n) \quad (\text{B.4})$$

where

$$\omega_i = \frac{d}{dt} \delta_i$$

The application of Runge-Kutta to a system of n equations representing an n -machine system is as follows, for the k^{th} machine:

First approximation: $\delta(k) = \delta_o(k),$

$$v(k) = v_o(k) = h\omega_o(k); \quad \text{at } t = t_o.$$

Second approximation:

$$\delta(k) = \delta_o(k) + \alpha v_o(k) + \beta k_1(k),$$

$$v(k) = v_o(k) + 2(\gamma k_1(k)); \quad \text{at } t = t_o + \alpha h.$$

Third approximation:

$$\delta(k) = \delta_o(k) + \alpha_1 v_o(k) + \beta_1 k_1(k) + \delta_1 k_2(k)$$

$$v(k) = v_o(k) + 2(\gamma_1 k_1(k) + \epsilon_1 k_2(k)); \quad \text{at } t = t_o + \alpha_1 h$$

Fourth approximation:

$$\delta(k) = \delta_o(k) + \alpha_2 v_o(k) + \beta_2 k_1(k) + \delta_2 k_2(k) + \xi_2 k_3(k)$$

$$v(k) = v_o(k) + 2(\gamma_2 k_1(k) + \epsilon_2 k_2(k) + \tau_2 k_3(k))$$

$$; \quad \text{at } t = t_o + \alpha_2 h$$

where

$$k_m(k) = \frac{h^2}{2} \left(\frac{d^2 \delta_k}{dt^2} \right)_m$$

and

$$\delta_1(k) = \delta_0(k) + h\nu_0(k) + \sum_{m=1}^4 \gamma_{0m} k_m(k);$$

$$\nu_1(k) = \nu_0(k) + \sum_{m=1}^4 \gamma_{1m} k_m(k) \cdot$$

The coefficients involved in the calculation of the $\langle k_{m(k)} \rangle$ depend on the particular Runge-Kutta variation used.

APPENDIX C

SOME COMPUTER RESULTS

KUTIA REPEATED--VARIOUS H--EQN. (3.21)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

C-1

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.586	27.592	27.594	27.594	27.594	27.594
0.125				32.522	32.523	32.523	32.523
0.150			38.173	38.177	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.544	50.830	50.923	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.420	64.429	64.430	64.430	64.430
0.275				71.086	71.087	71.087	71.087
0.300		77.368	77.527	77.537	77.538	77.538	77.538
0.325				83.712	83.713	83.713	83.713
0.350			89.560	89.571	89.572	89.572	89.572
0.375				95.099	95.100	95.100	95.100
0.400	98.317	100.093	100.294	100.306	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.868	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.396	118.672	118.687	118.688	118.688	118.688
0.525				122.976	122.977	122.977	122.977
0.550			127.285	127.303	127.304	127.304	127.304
0.575				131.768	131.769	131.769	131.769
0.600	130.847	136.016	136.461	136.484	136.486	136.486	136.486
0.625				140.661	140.662	140.662	140.662
0.650			143.557	143.593	143.595	143.595	143.595
0.675				145.486	145.488	145.488	145.487
0.700		145.336	146.423	146.474	146.477	146.477	146.476
0.725				146.631	146.634	146.634	146.634
0.750			145.891	145.968	145.972	145.972	145.971
0.775				144.437	144.441	144.441	144.441
0.800	104.877	139.269	141.805	141.925	141.931	141.931	141.930
0.825				138.257	138.264	138.264	138.263
0.850			132.990	133.184	133.193	133.193	133.191
0.875				126.387	126.398	126.398	126.396
0.900		110.657	117.176	117.486	117.499	117.499	117.497
0.925				106.072	106.089	106.089	106.087
0.950			91.314	91.786	91.806	91.806	91.803
0.975				74.448	74.473	74.472	74.469
1.000	-26.636	40.864	53.635	54.260	54.287	54.287	54.283

KUTTA REPEATED--VARIOUS H--EQN. (3.21)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.159	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.479	22.533	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.783	29.787	29.787	29.787	29.787
0.175				32.034	32.035	32.035	32.035
0.200	32.337	33.291	33.367	33.370	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.642	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.543	31.608	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.468	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	23.929	25.199	25.266	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.767	22.770	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.379	21.498	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.883	21.891	21.892	21.892	21.892
0.575				22.851	22.852	22.852	22.852
0.600	20.904	24.135	24.409	24.422	24.423	24.423	24.423
0.625				17.573	17.574	17.574	17.574
0.650			11.902	11.922	11.922	11.922	11.922
0.675				7.117	7.118	7.118	7.118
0.700		1.992	2.788	2.830	2.831	2.831	2.831
0.725				-1.249	-1.246	-1.246	-1.247
0.750			-5.501	-5.420	-5.416	-5.416	-5.417
0.775				-9.988	-9.983	-9.983	-9.983
0.800	-43.098	-18.377	-15.416	-15.273	-15.266	-15.266	-15.267
0.825				-21.618	-21.609	-21.610	-21.611
0.850			-29.617	-29.382	-29.371	-29.372	-29.373
0.875				-38.909	-38.896	-38.897	-38.899
0.900		-57.961	-50.795	-50.449	-50.434	-50.434	-50.436
0.925				-63.996	-63.979	-63.979	-63.982
0.950			-79.443	-79.032	-79.014	-79.015	-79.017
0.975				-94.191	-94.175	-94.176	-94.178
1.000	-69.208	-111.519	-107.323	-107.054	-107.043	-107.043	-107.045

GILL REPEATED--VARIOUS H--EQN. (3.23)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

C-2

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.568	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.582	27.592	27.594	27.594	27.594	27.594
0.125				32.522	32.523	32.523	32.523
0.150			38.172	38.177	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.251	50.819	50.922	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.419	64.429	64.430	64.430	64.430
0.275				71.086	71.087	71.087	71.087
0.300		77.348	77.526	77.537	77.538	77.538	77.538
0.325				83.712	83.713	83.713	83.713
0.350			89.559	89.571	89.572	89.572	89.572
0.375				95.099	95.100	95.100	95.100
0.400	97.598	100.067	100.293	100.306	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.866	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.360	118.670	118.687	118.688	118.688	118.688
0.525				122.976	122.977	122.977	122.977
0.550			127.283	127.303	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	129.362	135.959	136.459	136.484	136.486	136.486	136.486
0.625				140.661	140.662	140.662	140.662
0.650			143.553	143.593	143.595	143.595	143.595
0.675				145.485	145.488	145.488	145.487
0.700		145.187	146.416	146.474	146.477	146.477	146.476
0.725				146.631	146.634	146.634	146.634
0.750			145.879	145.968	145.972	145.972	145.971
0.775				144.436	144.441	144.441	144.441
0.800	100.074	138.820	141.783	141.924	141.931	141.931	141.930
0.825				138.256	138.264	138.264	138.263
0.850			132.953	133.182	133.192	133.192	133.191
0.875				126.384	126.398	126.398	126.396
0.900		109.404	117.113	117.482	117.499	117.499	117.497
0.925				106.068	106.089	106.089	106.087
0.950			91.215	91.780	91.806	91.806	91.803
0.975				74.441	74.472	74.472	74.469
1.000	-34.415	38.404	53.498	54.252	54.287	54.287	54.283

GILL REPEATED--VARIOUS H--EQN. (3.23)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.158	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.470	22.533	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.783	29.787	29.787	29.787	29.787
0.175				32.034	32.035	32.035	32.035
0.200	31.970	33.279	33.366	33.370	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.641	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.535	31.608	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.467	28.469	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	23.704	25.171	25.266	25.269	25.269	25.269	25.269
0.425				23.894	23.895	23.895	23.895
0.450			22.771	22.770	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.361	21.497	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.882	21.891	21.892	21.892	21.892
0.575				22.851	22.852	22.852	22.852
0.600	19.904	24.075	24.407	24.422	24.423	24.423	24.423
0.625				17.573	17.574	17.574	17.574
0.650			11.897	11.921	11.922	11.922	11.922
0.675				7.116	7.118	7.118	7.118
0.700		1.782	2.770	2.829	2.831	2.831	2.831
0.725				-1.250	-1.246	-1.246	-1.247
0.750			-5.519	-5.421	-5.416	-5.416	-5.417
0.775				-9.989	-9.983	-9.983	-9.983
0.800	-55.510	-18.983	-15.446	-15.274	-15.266	-15.266	-15.267
0.825				-21.620	-21.610	-21.610	-21.611
0.850			-29.667	-29.385	-29.372	-29.372	-29.373
0.875				-38.913	-38.897	-38.897	-38.899
0.900		-59.390	-50.872	-50.453	-50.434	-50.434	-50.436
0.925				-64.001	-63.979	-63.979	-63.982
0.950			-79.543	-79.038	-79.015	-79.015	-79.017
0.975				-94.147	-94.176	-94.176	-94.178
1.000	-50.567	-112.447	-107.402	-107.059	-107.043	-107.043	-107.045

STRACHEY REPEATED--VARIOUS H--EQN. (3.25)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

C-3

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.595	27.592	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.174	38.177	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	51.449	50.862	50.924	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.422	64.429	64.430	64.430	64.430
0.275				71.086	71.087	71.087	71.087
0.300		77.419	77.529	77.537	77.538	77.538	77.538
0.325				83.712	83.713	83.713	83.713
0.350			89.562	89.571	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	100.541	100.159	100.297	100.306	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.871	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.487	118.676	118.687	118.688	118.688	118.688
0.525				122.976	122.977	122.977	122.977
0.550			127.290	127.304	127.304	127.304	127.304
0.575				131.768	131.769	131.769	131.769
0.600	135.552	136.163	136.468	136.485	136.486	136.486	136.486
0.625				140.661	140.662	140.662	140.662
0.650			143.567	143.594	143.595	143.595	143.595
0.675				145.486	145.488	145.488	145.488
0.700		145.716	146.440	146.475	146.477	146.477	146.476
0.725				146.632	146.634	146.634	146.634
0.750			145.922	145.970	145.972	145.972	145.971
0.775				144.439	144.441	144.441	144.441
0.800	138.992	140.413	141.858	141.928	141.931	141.931	141.930
0.825				138.261	138.264	138.264	138.263
0.850			133.082	133.189	133.193	133.193	133.191
0.875				126.393	126.398	126.398	126.396
0.900		113.061	117.329	117.494	117.500	117.499	117.497
0.925				106.083	106.090	106.089	106.087
0.950			91.555	91.799	91.807	91.806	91.803
0.975				74.464	74.474	74.472	74.469
1.000	79.133	47.307	53.969	54.279	54.289	54.287	54.283

STRACHEY REPEATED--VARIOUS H--EQN. (3.25)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.154	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.501	22.534	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.785	29.787	29.787	29.787	29.787
0.175				32.034	32.035	32.035	32.035
0.200	33.468	33.323	33.368	33.370	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.643	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.562	31.609	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.460	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	24.647	25.221	25.267	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.769	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.425	21.500	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.886	21.891	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	24.182	24.237	24.413	24.422	24.423	24.423	24.423
0.625				17.574	17.574	17.574	17.574
0.650			11.916	11.922	11.922	11.922	11.922
0.675				7.118	7.118	7.118	7.118
0.700		2.926	2.813	2.831	2.832	2.831	2.831
0.725				-1.247	-1.246	-1.246	-1.247
0.750			-5.458	-5.418	-5.416	-5.416	-5.417
0.775				-9.985	-9.983	-9.983	-9.983
0.800	-3.057	-16.831	-15.344	-15.269	-15.266	-15.266	-15.267
0.825				-21.613	-21.609	-21.609	-21.611
0.850			-29.496	-29.375	-29.371	-29.372	-29.373
0.875				-38.901	-38.896	-38.897	-38.899
0.900		-54.261	-50.608	-50.438	-50.433	-50.434	-50.436
0.925				-63.984	-63.978	-63.979	-63.982
0.950			-79.197	-79.018	-79.013	-79.015	-79.017
0.975				-94.177	-94.174	-94.176	-94.178
1.000	-83.239	-108.802	-107.129	-107.043	-107.042	-107.043	-107.045

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

...VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.589	27.592	27.594	27.594	27.594	27.594
0.125				32.522	32.523	32.523	32.523
0.150			38.173	38.177	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.817	50.843	50.923	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.421	64.429	64.430	64.430	64.430
0.275				71.086	71.087	71.087	71.087
0.300		77.383	77.527	77.537	77.538	77.538	77.538
0.325				83.712	83.713	83.713	83.713
0.350			89.560	89.571	89.572	89.572	89.572
0.375				95.099	95.100	95.100	95.100
0.400	98.042	100.100	100.294	100.306	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.867	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		110.393	118.671	118.687	118.688	118.688	118.688
0.525				122.976	122.977	122.977	122.977
0.550			127.284	127.303	127.304	127.304	127.304
0.575				131.768	131.769	131.769	131.769
0.600	130.003	136.004	136.460	136.484	136.486	136.486	136.486
0.625				140.661	140.662	140.662	140.662
0.650			143.553	143.593	143.595	143.595	143.595
0.675				145.485	145.488	145.488	145.487
0.700		145.267	146.416	146.474	146.477	146.477	146.476
0.725				146.630	146.634	146.634	146.634
0.750			145.800	145.967	145.972	145.972	145.971
0.775				144.436	144.441	144.441	144.441
0.800	105.537	139.115	141.785	141.924	141.931	141.931	141.930
0.825				138.255	138.264	138.264	138.263
0.850			132.957	133.182	133.192	133.192	133.191
0.875				126.384	126.390	126.398	126.396
0.900		110.296	117.124	117.482	117.499	117.499	117.497
0.925				106.068	106.089	106.089	106.087
0.950			91.238	91.781	91.806	91.806	91.803
0.975				74.442	74.472	74.472	74.469
1.000	-19.877	40.474	53.540	54.253	54.287	54.287	54.283

BOULTON REPEATED--VARIOUS H--EON. (3.27)

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

...VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.150	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.470	22.533	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.783	29.787	29.787	29.787	29.787
0.175				32.034	32.035	32.035	32.035
0.200	31.887	33.287	33.367	33.370	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.642	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.544	31.608	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.467	28.469	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	23.794	25.195	25.266	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.767	22.770	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.371	21.497	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.882	21.891	21.892	21.892	21.892
0.575				22.851	22.852	22.852	22.852
0.600	20.262	24.119	24.407	24.422	24.423	24.423	24.423
0.625				17.573	17.574	17.574	17.574
0.650			11.898	11.921	11.922	11.922	11.922
0.675				7.116	7.118	7.118	7.118
0.700		1.944	2.780	2.829	2.831	2.831	2.831
0.725				-1.250	-1.246	-1.246	-1.247
0.750			-5.517	-5.421	-5.416	-5.416	-5.417
0.775				-9.989	-9.983	-9.983	-9.983
0.800	-45.625	-18.581	-15.444	-15.275	-15.266	-15.266	-15.267
0.825				-21.620	-21.610	-21.610	-21.611
0.850			-29.663	-29.385	-29.372	-29.372	-29.373
0.875				-38.913	-38.897	-38.897	-38.899
0.900		-58.545	-50.869	-50.454	-50.434	-50.434	-50.436
0.925				-64.002	-63.979	-63.980	-63.982
0.950			-79.540	-79.038	-79.015	-79.015	-79.017
0.975				-94.198	-94.176	-94.176	-94.178
1.000	-44.419	-111.726	-107.383	-107.059	-107.043	-107.043	-107.045

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.568	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.582	27.593	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.176	38.178	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.196	50.094	50.929	50.931	50.931	50.931	50.931
0.225				57.668	57.668	57.668	57.668
0.250			64.428	64.430	64.430	64.430	64.430
0.275				71.087	71.087	71.087	71.087
0.300		77.500	77.536	77.538	77.538	77.538	77.538
0.325				83.713	83.713	83.713	83.713
0.350			89.570	89.572	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	99.550	100.281	100.306	100.307	100.307	100.307	100.307
0.425				105.220	105.220	105.220	105.220
0.450			109.881	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.663	118.688	118.688	118.688	118.688	118.688
0.525				122.977	122.977	122.977	122.977
0.550			127.304	127.304	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	135.127	136.458	136.486	136.486	136.486	136.486	136.486
0.625				140.662	140.662	140.662	140.662
0.650			143.595	143.595	143.595	143.595	143.595
0.675				145.488	145.488	145.488	145.487
0.700		146.439	146.478	146.477	146.477	146.477	146.476
0.725				146.635	146.634	146.634	146.634
0.750			145.975	145.972	145.972	145.972	145.971
0.775				144.442	144.441	144.441	144.441
0.800	134.919	141.840	141.937	141.932	141.931	141.931	141.930
0.825				138.266	138.264	138.264	138.263
0.850			133.204	133.195	133.193	133.193	133.191
0.875				126.400	126.398	126.398	126.396
0.900		117.260	117.519	117.502	117.500	117.499	117.497
0.925				106.093	106.090	106.089	106.087
0.950			91.832	91.811	91.808	91.806	91.803
0.975				74.470	74.474	74.472	74.469
1.000	11.412	53.581	54.319	54.293	54.289	54.287	54.283

MERSON REPEATED--VARIOUS H--EQN. (3.34)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.160	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.528	22.536	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.787	29.787	29.787	29.787	29.787
0.175				32.035	32.035	32.035	32.035
0.200	33.151	33.369	33.371	33.371	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.645	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.618	31.611	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.470	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	25.131	25.274	25.269	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.771	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.501	21.503	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.892	21.892	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	23.444	24.408	24.423	24.423	24.423	24.423	24.423
0.625				17.574	17.574	17.574	17.574
0.650			11.924	11.923	11.922	11.922	11.922
0.675				7.118	7.118	7.118	7.118
0.700		2.782	2.835	2.832	2.832	2.831	2.831
0.725				-1.246	-1.246	-1.246	-1.247
0.750			-5.411	-5.415	-5.416	-5.416	-5.417
0.775				-9.982	-9.982	-9.983	-9.983
0.800	-26.002	-15.377	-15.256	-15.265	-15.266	-15.266	-15.267
0.825				-21.607	-21.609	-21.609	-21.611
0.850			-29.353	-29.369	-29.371	-29.372	-29.373
0.875				-38.893	-38.896	-38.897	-38.899
0.900		-50.630	-50.404	-50.430	-50.433	-50.434	-50.436
0.925				-63.974	-63.978	-63.979	-63.982
0.950			-78.975	-79.009	-79.013	-79.015	-79.017
0.975				-94.170	-94.174	-94.176	-94.178
1.000	-108.254	-106.998	-107.010	-107.039	-107.042	-107.043	-107.045

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....ERROR IN DELTA ON THIS PAGE, VALUES*1000.

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	7200.042	-96.411	-18.716	-1.425	-0.083	-0.005
0.025				-0.000	-0.000	-0.000	-0.000
0.050			-0.006	-0.004	-0.000	-0.000	-0.000
0.075				-0.010	-0.001	-0.000	-0.000
0.100		-0.426	-0.257	-0.018	-0.001	-0.000	-0.000
0.125				-0.029	-0.002	-0.000	-0.000
0.150			-0.623	-0.041	-0.003	-0.000	-0.000
0.175				-0.054	-0.003	-0.000	-0.000
0.200	-32.050	-15.089	-1.032	-0.066	-0.004	-0.000	-0.000
0.225				-0.075	-0.005	-0.000	-0.000
0.250			-1.310	-0.081	-0.005	-0.000	-0.000
0.275				-0.082	-0.005	-0.000	-0.000
0.300		-23.592	-1.344	-0.080	-0.005	-0.000	-0.000
0.325				-0.075	-0.005	-0.000	-0.000
0.350			-1.180	-0.068	-0.004	-0.000	-0.000
0.375				-0.061	-0.004	-0.000	-0.000
0.400	-410.015	-18.077	-0.948	-0.054	-0.003	-0.000	-0.000
0.425				-0.048	-0.003	-0.000	-0.000
0.450			-0.748	-0.042	-0.003	-0.000	-0.000
0.475				-0.038	-0.002	-0.000	-0.000
0.500		-11.774	-0.619	-0.035	-0.002	-0.000	-0.000
0.525				-0.033	-0.002	-0.000	-0.000
0.550			-0.569	-0.033	-0.002	-0.000	-0.000
0.575				-0.033	-0.002	-0.000	-0.000
0.600	-195.456	-10.575	-0.601	-0.035	-0.002	-0.000	-0.000
0.625				-0.088	-0.004	-0.000	-0.000
0.650			-1.752	-0.051	-0.002	-0.000	-0.000
0.675				-0.028	-0.001	-0.000	-0.000
0.700		-41.893	-0.695	-0.014	-0.000	-0.000	-0.000
0.725				-0.005	-0.000	-0.000	-0.000
0.750			-0.141	0.001	0.000	-0.000	-0.000
0.775				0.004	-0.000	-0.000	-0.000
0.800	-1124.845	-4.573	0.156	0.001	-0.001	-0.000	-0.000
0.825				-0.010	-0.002	-0.000	-0.000
0.850			0.021	-0.039	-0.004	-0.000	-0.000
0.875				-0.106	-0.009	-0.001	-0.000
0.900		3.756	-1.621	-0.219	-0.018	-0.001	-0.000
0.925				-0.429	-0.034	-0.002	-0.000
0.950			-7.849	-0.758	-0.057	-0.004	-0.000
0.975				-1.172	-0.083	-0.005	-0.000
1.000	-1023.836	-172.882	-20.314	-1.498	-0.098	-0.006	-0.000

MERSON REPEATED--VARIOUS H--EUN. (3.34)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....ERROR IN OMEGA ON THIS PAGE, VALUES*1000.

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	-4904.946	1518.484	66.854	2.000	0.201	0.015
0.025				-0.017	-0.001	-0.000	0.000
0.050			-0.541	-0.027	-0.001	-0.000	-0.000
0.075				-0.037	-0.002	-0.000	-0.000
0.100		-17.405	-0.857	-0.048	-0.003	-0.000	-0.000
0.125				-0.057	-0.004	-0.000	-0.000
0.150			-1.077	-0.062	-0.004	-0.000	-0.000
0.175				-0.059	-0.003	-0.000	-0.000
0.200	-523.117	-20.087	-0.915	-0.048	-0.003	-0.000	-0.000
0.225				-0.031	-0.002	-0.000	-0.000
0.250			-0.352	-0.012	-0.000	-0.000	-0.000
0.275				0.007	0.001	0.000	0.000
0.300		-0.468	0.244	0.022	0.002	0.000	0.000
0.325				0.031	0.002	0.000	0.000
0.350			0.554	0.036	0.002	0.000	0.000
0.375				0.036	0.002	0.000	0.000
0.400	131.565	9.731	0.568	0.033	0.002	0.000	0.000
0.425				0.029	0.002	0.000	0.000
0.450			0.425	0.023	0.001	0.000	0.000
0.475				0.017	0.001	0.000	0.000
0.500		5.729	0.238	0.011	0.001	0.000	0.000
0.525				0.005	0.000	0.000	0.000
0.550			0.044	-0.001	-0.000	-0.000	0.000
0.575				-0.007	-0.001	-0.000	0.000
0.600	70.974	-0.531	-0.170	-0.014	-0.001	-0.000	-0.000
0.625				0.223	0.011	0.001	0.000
0.650			3.677	0.135	0.006	0.000	0.000
0.675				0.084	0.004	0.000	0.000
0.700		72.954	1.748	0.054	0.002	0.000	-0.000
0.725				0.035	0.001	0.000	0.000
0.750			0.937	0.018	-0.000	-0.000	0.000
0.775				-0.002	-0.002	-0.000	-0.000
0.800	1798.509	26.128	0.219	-0.035	-0.004	-0.000	-0.000
0.825				-0.097	-0.009	-0.001	-0.000
0.850			-1.797	-0.213	-0.017	-0.001	-0.000
0.875				-0.426	-0.033	-0.002	-0.000
0.900		-57.769	-8.812	-0.789	-0.058	-0.004	-0.000
0.925				-1.327	-0.093	-0.006	-0.000
0.950			-25.828	-1.916	-0.127	-0.008	-0.001
0.975				-2.107	-0.125	-0.007	-0.000
1.000	-7323.585	-539.591	-29.530	-1.178	-0.044	-0.002	-0.000

EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS. AE = 0.10E 00

TIME, SEC	VALUES OF DELTA	VALUES OF OMEGA	STEP SIZE
0.000	18.105	0.000	0.050
0.025	*****	*****	*****
0.050	20.568	12.160	0.100
0.075	*****	*****	*****
0.100	*****	*****	*****
0.125	*****	*****	*****
0.150	38.159	29.783	0.100
0.175	*****	*****	*****
0.200	*****	*****	*****
0.225	*****	*****	*****
0.250	64.397	33.648	0.100
0.275	*****	*****	*****
0.300	*****	*****	*****
0.325	*****	*****	*****
0.350	89.546	28.475	0.100
0.375	*****	*****	*****
0.400	*****	*****	*****
0.425	*****	*****	*****
0.450	109.864	22.772	0.100
0.475	*****	*****	*****
0.500	*****	*****	*****
0.525	*****	*****	*****
0.550	127.286	21.885	0.100
0.575	*****	*****	*****
0.600	136.464	24.411	0.100
0.625	*****	*****	*****
0.650	*****	*****	*****
0.675	*****	*****	*****
0.700	146.452	2.798	0.050
0.725	*****	*****	*****
0.750	145.930	-5.473	0.100
0.775	*****	*****	*****
0.800	*****	*****	*****
0.825	*****	*****	*****
0.850	133.093	-29.473	0.100
0.875	*****	*****	*****
0.900	*****	*****	*****
0.925	*****	*****	*****
0.950	91.487	-79.074	0.050
0.975	*****	*****	*****
1.000	53.945	-107.021	0.050

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.601	27.595	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.182	38.178	38.178	38.178	38.178
0.175				44.377	44.376	44.375	44.375
0.200	51.315	50.986	50.942	50.932	50.931	50.931	50.931
0.225				57.670	57.668	57.668	57.668
0.250			64.450	64.433	64.430	64.430	64.430
0.275				71.070	71.087	71.087	71.087
0.300		77.717	77.567	77.542	77.538	77.538	77.538
0.325				83.718	83.714	83.713	83.713
0.350			89.610	89.577	89.572	89.572	89.572
0.375				95.106	95.101	95.100	95.100
0.400	102.442	100.622	100.354	100.313	100.308	100.307	100.307
0.425				105.226	105.220	105.220	105.220
0.450			109.938	109.889	109.882	109.881	109.881
0.475				114.357	114.349	114.349	114.348
0.500		119.177	118.759	118.697	118.689	118.688	118.688
0.525				122.988	122.978	122.977	122.977
0.550			127.395	127.316	127.306	127.304	127.304
0.575				131.783	131.771	131.769	131.769
0.600	141.646	137.332	136.607	136.502	136.488	136.486	136.486
0.625				140.681	140.665	140.663	140.662
0.650			143.776	143.619	143.598	143.595	143.595
0.675				145.518	145.492	145.488	145.488
0.700		148.546	146.781	146.517	146.482	146.477	146.476
0.725				146.687	146.641	146.635	146.634
0.750			146.498	146.041	145.981	145.973	145.972
0.775				144.531	144.453	144.443	144.441
0.800	180.694	148.371	142.835	142.049	141.946	141.933	141.930
0.825				138.417	138.284	138.266	138.263
0.850			134.723	133.391	133.210	133.196	133.192
0.875				126.653	126.430	126.402	126.396
0.900		136.156	120.012	117.824	117.541	117.504	117.498
0.925				106.496	106.141	106.095	106.087
0.950			95.682	92.304	91.870	91.814	91.804
0.975				75.063	74.548	74.481	74.470
1.000	433.669	37.789	59.540	54.956	54.372	54.297	54.284

JOHNSON-WARD METHOD--VARIOUS H--EQN. (5.12)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.163	12.161	12.161	12.161	12.161
0.075				17.672	17.671	17.671	17.671
0.100		22.598	22.546	22.538	22.537	22.537	22.537
0.125				26.610	26.609	26.608	26.608
0.150			29.805	29.789	29.788	29.787	29.787
0.175				32.037	32.035	32.035	32.035
0.200	34.618	33.541	33.194	33.374	33.371	33.371	33.371
0.225				33.872	33.869	33.869	33.869
0.250			33.669	33.648	33.645	33.645	33.645
0.275				32.844	32.841	32.841	32.841
0.300		31.779	31.633	31.613	31.611	31.611	31.611
0.325				30.109	30.107	30.107	30.107
0.350			28.491	28.472	28.470	28.470	28.470
0.375				26.825	26.823	26.822	26.822
0.400	26.089	25.435	25.291	25.271	25.269	25.269	25.269
0.425				23.898	23.895	23.895	23.895
0.450			22.800	22.774	22.771	22.771	22.771
0.475				21.960	21.956	21.956	21.956
0.500		21.808	21.545	21.508	21.504	21.503	21.503
0.525				21.470	21.464	21.463	21.463
0.550			21.955	21.900	21.893	21.892	21.892
0.575				22.862	22.853	22.852	22.852
0.600	29.200	25.128	24.520	24.435	24.424	24.423	24.423
0.625				17.594	17.576	17.574	17.574
0.650			12.152	11.952	11.926	11.923	11.922
0.675				7.158	7.123	7.119	7.118
0.700		5.949	3.249	2.885	2.830	2.832	2.831
0.725				-1.176	-1.237	-1.245	-1.247
0.750			-4.699	-5.324	-5.404	-5.415	-5.417
0.775				-9.862	-9.967	-9.981	-9.983
0.800	55.578	-6.360	-14.062	-15.111	-15.246	-15.264	-15.267
0.825				-21.410	-21.584	-21.606	-21.610
0.850			-27.411	-29.119	-29.339	-29.368	-29.373
0.875				-38.582	-38.856	-38.892	-38.898
0.900		-27.214	-47.466	-50.054	-50.385	-50.428	-50.436
0.925				-63.540	-63.923	-63.973	-63.981
0.950			-75.276	-78.544	-78.955	-79.007	-79.017
0.975				-93.732	-94.119	-94.169	-94.178
1.000	235.553	-73.798	-104.320	-106.721	-107.002	-107.038	-107.044

COLLATZ PROCEDURE--VARIOUS H--EON. (5.17)

C-9

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.568	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.582	27.593	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.175	38.178	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.196	50.881	50.927	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.426	64.430	64.430	64.430	64.430
0.275				71.087	71.087	71.087	71.087
0.300		77.482	77.534	77.538	77.538	77.538	77.538
0.325				83.713	83.713	83.713	83.713
0.350			89.568	89.572	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	100.261	100.256	100.304	100.307	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.879	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.628	118.685	118.688	118.688	118.688	118.688
0.525				122.977	122.977	122.977	122.977
0.550			127.301	127.304	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	136.240	136.306	136.481	136.486	136.486	136.486	136.486
0.625				140.662	140.662	140.662	140.662
0.650			143.586	143.595	143.595	143.595	143.595
0.675				145.487	145.488	145.488	145.487
0.700		146.143	146.461	146.476	146.477	146.477	146.476
0.725				146.633	146.634	146.634	146.634
0.750			145.948	145.971	145.972	145.972	145.971
0.775				144.440	144.441	144.441	144.441
0.800	137.499	141.089	141.093	141.930	141.931	141.931	141.930
0.825				138.263	138.264	138.264	138.263
0.850			133.131	133.191	133.193	133.193	133.191
0.875				126.395	126.398	126.398	126.396
0.900		115.254	117.399	117.496	117.500	117.499	117.497
0.925				106.085	106.090	106.089	106.087
0.950			91.648	91.801	91.807	91.806	91.803
0.975				74.466	74.474	74.472	74.469
1.000	30.438	49.436	54.073	54.280	54.289	54.287	54.283

COLLATZ PROCEDURE--VARIOUS H--EON. (5.17)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.161	12.161	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.557	22.538	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.789	29.787	29.787	29.787	29.787
0.175				32.035	32.035	32.035	32.035
0.200	33.818	33.384	33.372	33.371	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.646	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.609	31.611	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.470	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	25.281	25.264	25.269	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.771	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.492	21.503	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.892	21.892	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	24.649	24.389	24.422	24.423	24.423	24.423	24.423
0.625				17.574	17.574	17.574	17.574
0.650			11.918	11.922	11.922	11.922	11.922
0.675				7.118	7.118	7.118	7.118
0.700		2.527	2.818	2.831	2.832	2.831	2.831
0.725				-1.247	-1.246	-1.246	-1.247
0.750			-5.445	-5.417	-5.416	-5.416	-5.417
0.775				-9.984	-9.983	-9.983	-9.983
0.800	-20.112	-16.304	-15.318	-15.268	-15.266	-15.266	-15.267
0.825				-21.612	-21.609	-21.609	-21.611
0.850			-29.455	-29.374	-29.371	-29.372	-29.373
0.875				-38.900	-38.896	-38.897	-38.899
0.900		-53.038	-50.551	-50.437	-50.433	-50.434	-50.436
0.925				-63.983	-63.978	-63.979	-63.982
0.950			-79.146	-79.018	-79.013	-79.015	-79.017
0.975				-94.179	-94.175	-94.176	-94.178
1.000	-100.361	-108.824	-107.144	-107.046	-107.042	-107.043	-107.045

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.594	27.594	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.178	38.178	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.945	50.932	50.931	50.931	50.931	50.931	50.931
0.225				57.668	57.668	57.668	57.668
0.250			64.430	64.430	64.430	64.430	64.430
0.275				71.087	71.087	71.087	71.087
0.300		77.549	77.538	77.538	77.538	77.538	77.538
0.325				83.713	83.713	83.713	83.713
0.350			89.573	89.572	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	101.123	100.333	100.308	100.307	100.307	100.307	100.307
0.425				105.220	105.220	105.220	105.220
0.450			109.883	109.881	109.881	109.881	109.881
0.475				114.349	114.348	114.348	114.348
0.500		118.729	118.690	118.688	118.688	118.688	118.688
0.525				122.977	122.977	122.977	122.977
0.550			127.307	127.304	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	138.571	136.552	136.489	136.486	136.486	136.486	136.486
0.625				140.663	140.662	140.662	140.662
0.650			143.600	143.595	143.595	143.595	143.595
0.675				145.488	145.488	145.488	145.487
0.700		146.606	146.404	146.477	146.477	146.477	146.476
0.725				146.635	146.634	146.634	146.634
0.750			145.983	145.973	145.972	145.972	145.971
0.775				144.442	144.441	144.441	144.441
0.800	153.938	142.262	141.948	141.932	141.931	141.931	141.930
0.825				138.266	138.265	138.264	138.263
0.850			133.221	133.195	133.193	133.193	133.191
0.875				126.401	126.398	126.398	126.396
0.900		118.365	117.544	117.503	117.500	117.499	117.497
0.925				106.094	106.090	106.089	106.087
0.950			91.869	91.812	91.808	91.806	91.803
0.975				74.479	74.474	74.472	74.469
1.000	143.691	55.841	54.367	54.294	54.289	54.287	54.283

SECOND ORDER METHOD--VARIOUS H--EQN. (5.25)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

....VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.161	12.161	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.557	22.538	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.789	29.787	29.787	29.787	29.787
0.175				32.035	32.035	32.035	32.035
0.200	33.933	33.493	33.372	33.371	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.646	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.622	31.611	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.470	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	25.662	25.280	25.269	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.771	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.527	21.504	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.893	21.892	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	26.029	24.477	24.426	24.423	24.423	24.423	24.423
0.625				17.574	17.574	17.574	17.574
0.650			11.927	11.923	11.922	11.922	11.922
0.675				7.118	7.118	7.118	7.118
0.700		2.967	2.839	2.832	2.832	2.831	2.831
0.725				-1.246	-1.246	-1.246	-1.247
0.750			-5.404	-5.415	-5.416	-5.416	-5.417
0.775				-9.981	-9.982	-9.983	-9.983
0.800	0.659	-14.865	-15.245	-15.264	-15.266	-15.266	-15.267
0.825				-21.607	-21.609	-21.609	-21.611
0.850			-29.337	-29.369	-29.371	-29.372	-29.373
0.875				-38.893	-38.896	-38.897	-38.899
0.900		-49.382	-50.379	-50.429	-50.433	-50.434	-50.436
0.925				-63.974	-63.978	-63.979	-63.982
0.950			-78.942	-79.009	-79.013	-79.015	-79.017
0.975				-94.170	-94.174	-94.176	-94.178
1.000	-18.605	-106.045	-106.999	-107.039	-107.042	-107.043	-107.045

SECOND ORDER METHOD--VARIOUS H--EQN. (5.46)

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

C-11

...VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.569	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.591	27.594	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.177	38.178	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	50.784	50.920	50.930	50.931	50.931	50.931	50.931
0.225				57.668	57.668	57.668	57.668
0.250			64.429	64.430	64.430	64.430	64.430
0.275				71.087	71.087	71.087	71.087
0.300		77.528	77.537	77.538	77.538	77.538	77.538
0.325				83.713	83.713	83.713	83.713
0.350			89.571	89.572	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	100.316	100.302	100.307	100.307	100.307	100.307	100.307
0.425				105.220	105.220	105.220	105.220
0.450			109.881	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.687	118.688	118.688	118.688	118.688	118.688
0.525				122.977	122.977	122.977	122.977
0.550			127.304	127.304	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	136.590	136.485	136.486	136.486	136.486	136.486	136.486
0.625				140.662	140.662	140.662	140.662
0.650			143.595	143.595	143.595	143.595	143.595
0.675				145.488	145.488	145.488	145.488
0.700		146.459	146.477	146.477	146.477	146.477	146.476
0.725				146.634	146.634	146.634	146.634
0.750			145.972	145.972	145.972	145.972	145.971
0.775				144.442	144.441	144.441	144.441
0.800	141.798	141.892	141.932	141.932	141.931	141.931	141.930
0.825				138.265	138.264	138.264	138.263
0.850			133.194	133.193	133.193	133.193	133.191
0.875				126.399	126.398	126.398	126.396
0.900		117.397	117.502	117.501	117.500	117.499	117.497
0.925				106.091	106.090	106.089	106.087
0.950			91.808	91.809	91.808	91.806	91.803
0.975				74.475	74.474	74.472	74.469
1.000	53.011	53.984	54.288	54.290	54.289	54.287	54.283

SECOND ORDER METHOD--VARIOUS H--EQN. (5.46)

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

...VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.161	12.161	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.541	22.537	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.788	29.787	29.787	29.787	29.787
0.175				32.035	32.035	32.035	32.035
0.200	33.504	33.376	33.371	33.371	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.645	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.615	31.611	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.470	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	25.316	25.277	25.269	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.771	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.507	21.503	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.892	21.892	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	24.562	24.428	24.423	24.423	24.423	24.423	24.423
0.625				17.574	17.574	17.574	17.574
0.650			11.923	11.923	11.922	11.922	11.922
0.675				7.118	7.118	7.118	7.118
0.700		2.822	2.832	2.832	2.831	2.831	2.831
0.725				-1.246	-1.246	-1.246	-1.247
0.750			-5.416	-5.416	-5.416	-5.416	-5.417
0.775				-9.982	-9.982	-9.983	-9.983
0.800	-15.363	-15.319	-15.265	-15.265	-15.266	-15.266	-15.267
0.825				-21.608	-21.609	-21.609	-21.611
0.850			-29.370	-29.370	-29.371	-29.372	-29.373
0.875				-38.895	-38.896	-38.897	-38.899
0.900		-50.537	-50.429	-50.432	-50.433	-50.434	-50.436
0.925				-63.977	-63.978	-63.979	-63.982
0.950			-79.007	-79.012	-79.013	-79.015	-79.017
0.975				-94.173	-94.174	-94.176	-94.178
1.000	-105.488	-107.079	-107.039	-107.041	-107.042	-107.043	-107.045

NEW AUTOMATIC PROCEDURE, EQN. (6.17)

EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS. AE = 0.10E 02

TIME, SEC	VALUES OF DELTA	VALUES OF OMEGA	STEP SIZE
0.000	18.105	0.000	0.050
0.025	*****	*****	*****
0.050	20.569	12.158	0.100
0.075	*****	*****	*****
0.100	*****	*****	*****
0.125	*****	*****	*****
0.150	38.178	29.738	0.100
0.175	*****	*****	*****
0.200	*****	*****	*****
0.225	*****	*****	*****
0.250	64.438	33.722	0.100
0.275	*****	*****	*****
0.300	*****	*****	*****
0.325	*****	*****	*****
0.350	89.642	28.568	0.200
0.375	*****	*****	*****
0.400	*****	*****	*****
0.425	*****	*****	*****
0.450	*****	*****	*****
0.475	*****	*****	*****
0.500	*****	*****	*****
0.525	*****	*****	*****
0.550	127.451	21.453	0.200
0.575	*****	*****	*****
0.600	136.461	23.998	0.100
0.625	*****	*****	*****
0.650	*****	*****	*****
0.675	*****	*****	*****
0.700	146.026	2.081	0.100
0.725	*****	*****	*****
0.750	*****	*****	*****
0.775	*****	*****	*****
0.800	140.488	-17.072	0.100
0.825	*****	*****	*****
0.850	*****	*****	*****
0.875	*****	*****	*****
0.900	113.624	-54.849	0.100
0.925	*****	*****	*****
0.950	*****	*****	*****
0.975	*****	*****	*****
1.000	46.591	-110.097	0.100

NEW AUTOMATIC PROCEDURE, EQN. (6.18)

EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS. AE = 0.10E 02

TIME, SEC	VALUES OF DELTA	VALUES OF OMEGA	STEP SIZE
0.000	18.105	0.000	0.050
0.025	*****	*****	*****
0.050	20.569	12.161	0.100
0.075	*****	*****	*****
0.100	*****	*****	*****
0.125	*****	*****	*****
0.150	38.169	29.792	0.200
0.175	*****	*****	*****
0.200	*****	*****	*****
0.225	*****	*****	*****
0.250	*****	*****	*****
0.275	*****	*****	*****
0.300	*****	*****	*****
0.325	*****	*****	*****
0.350	89.326	28.346	0.200
0.375	*****	*****	*****
0.400	*****	*****	*****
0.425	*****	*****	*****
0.450	*****	*****	*****
0.475	*****	*****	*****
0.500	*****	*****	*****
0.525	*****	*****	*****
0.550	126.800	21.602	0.200
0.575	*****	*****	*****
0.600	136.199	24.893	0.100
0.625	*****	*****	*****
0.650	*****	*****	*****
0.675	*****	*****	*****
0.700	146.447	3.100	0.200
0.725	*****	*****	*****
0.750	*****	*****	*****
0.775	*****	*****	*****
0.800	*****	*****	*****
0.825	*****	*****	*****
0.850	*****	*****	*****
0.875	*****	*****	*****
0.900	118.155	-49.564	0.200
0.925	*****	*****	*****
0.950	*****	*****	*****
0.975	*****	*****	*****
1.000	55.623	-106.231	0.200

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

...VALUES OF DELTA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	18.105	18.105	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727	18.727	18.727
0.050			20.568	20.569	20.569	20.569	20.569
0.075				23.561	23.561	23.561	23.561
0.100		27.569	27.593	27.594	27.594	27.594	27.594
0.125				32.523	32.523	32.523	32.523
0.150			38.175	38.178	38.178	38.178	38.178
0.175				44.375	44.375	44.375	44.375
0.200	49.463	50.856	50.926	50.930	50.931	50.931	50.931
0.225				57.667	57.668	57.668	57.668
0.250			64.425	64.430	64.430	64.430	64.430
0.275				71.087	71.087	71.087	71.087
0.300		77.442	77.533	77.538	77.538	77.538	77.538
0.325				83.713	83.713	83.713	83.713
0.350			89.567	89.571	89.572	89.572	89.572
0.375				95.100	95.100	95.100	95.100
0.400	98.010	100.212	100.303	100.307	100.307	100.307	100.307
0.425				105.219	105.220	105.220	105.220
0.450			109.878	109.881	109.881	109.881	109.881
0.475				114.348	114.348	114.348	114.348
0.500		118.579	118.684	118.688	118.688	118.688	118.688
0.525				122.977	122.977	122.977	122.977
0.550			127.300	127.304	127.304	127.304	127.304
0.575				131.769	131.769	131.769	131.769
0.600	132.020	136.317	136.480	136.485	136.486	136.486	136.486
0.625				140.660	140.662	140.662	140.662
0.650			143.556	143.589	143.594	143.595	143.595
0.675				145.478	145.486	145.487	145.487
0.700		145.550	146.363	146.462	146.475	146.476	146.476
0.725				146.613	146.632	146.634	146.634
0.750			145.752	145.944	145.968	145.971	145.971
0.775				144.404	144.437	144.441	144.441
0.800	107.157	138.606	141.542	141.881	141.925	141.930	141.930
0.825				138.199	138.256	138.263	138.263
0.850			132.526	133.108	133.182	133.191	133.191
0.875				126.288	126.384	126.396	126.396
0.900		108.221	116.399	117.359	117.482	117.497	117.497
0.925				105.913	106.067	106.086	106.086
0.950			90.106	91.590	91.780	91.803	91.803
0.975				74.215	74.441	74.468	74.468
1.000	-37.090	36.058	52.010	53.995	54.251	54.282	54.282

DEVOGELAERE METHOD--VARIOUS H--EQN. A.1-.5

...EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

...VALUES OF OMEGA ON THIS PAGE

TIME	H=0.200	0.100	0.050	0.025	0.0125	0.00625	.003125
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.025				6.197	6.197	6.197	6.197
0.050			12.159	12.160	12.161	12.161	12.161
0.075				17.671	17.671	17.671	17.671
0.100		22.498	22.535	22.537	22.537	22.537	22.537
0.125				26.608	26.608	26.608	26.608
0.150			29.785	29.787	29.787	29.787	29.787
0.175				32.035	32.035	32.035	32.035
0.200	37.334	33.331	33.369	33.371	33.371	33.371	33.371
0.225				33.869	33.869	33.869	33.869
0.250			33.644	33.645	33.645	33.645	33.645
0.275				32.841	32.841	32.841	32.841
0.300		31.602	31.611	31.611	31.611	31.611	31.611
0.325				30.107	30.107	30.107	30.107
0.350			28.470	28.470	28.470	28.470	28.470
0.375				26.822	26.822	26.822	26.822
0.400	24.832	25.261	25.269	25.269	25.269	25.269	25.269
0.425				23.895	23.895	23.895	23.895
0.450			22.771	22.771	22.771	22.771	22.771
0.475				21.956	21.956	21.956	21.956
0.500		21.464	21.502	21.503	21.503	21.503	21.503
0.525				21.463	21.463	21.463	21.463
0.550			21.890	21.892	21.892	21.892	21.892
0.575				22.852	22.852	22.852	22.852
0.600	21.459	24.314	24.420	24.423	24.423	24.423	24.423
0.625				17.555	17.572	17.573	17.574
0.650			11.756	11.903	11.920	11.922	11.922
0.675				7.096	7.115	7.118	7.118
0.700		0.869	2.617	2.805	2.828	2.831	2.831
0.725				-1.279	-1.250	-1.247	-1.247
0.750			-5.746	-5.458	-5.421	-5.417	-5.417
0.775				-10.036	-9.989	-9.984	-9.984
0.800	-59.688	-19.835	-15.804	-15.335	-15.275	-15.267	-15.267
0.825				-21.697	-21.620	-21.611	-21.611
0.850			-30.240	-29.482	-29.385	-29.373	-29.373
0.875				-39.035	-38.914	-38.899	-38.899
0.900		-60.979	-51.734	-50.600	-50.454	-50.437	-50.436
0.925				-64.172	-64.003	-63.983	-63.982
0.950			-80.604	-79.220	-79.040	-79.018	-79.018
0.975				-94.368	-94.199	-94.179	-94.179
1.000	-68.208	-113.354	-108.099	-107.183	-107.060	-107.045	-107.045

GILL REPEATED--VARIOUS H--EQN. (3.23)

.....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

TIME, SEC	VALUES OF DELTA FOR VARIOUS H BEGINNING AT H = 0.200.			
0.000	18.105	18.105	18.105	18.105
0.025				18.727
0.050			20.568	20.569
0.075				23.561
0.100		27.582	27.592	27.594
0.125				32.522
0.150			38.172	38.177
0.175				44.375
0.200	50.251	50.819	50.922	50.930
0.225				57.667
0.250			64.419	64.429
0.275				71.086
0.300		77.348	77.526	77.537
0.325				83.712
0.350			89.559	89.571
0.375				95.099
0.400	97.598	100.067	100.293	100.306
0.425				105.219
0.450			109.866	109.881
0.475				114.348
0.500		118.360	118.670	118.687
0.525				122.976
0.550			127.283	127.303
0.575				131.768
0.600	129.362	135.959	136.459	136.484
0.625				140.661
0.650			143.553	143.593
0.675				145.485
0.700		145.187	146.416	146.474
0.725				146.631
0.750			145.879	145.968
0.775				144.436
0.800	100.074	138.820	141.783	141.924
0.825				138.256
0.850			132.953	133.182
0.875				126.384
0.900		109.404	117.113	117.482
0.925				106.068
0.950			91.215	91.780
0.975				74.441
1.000	-34.415	38.404	53.498	54.252

.....FOR ALLOWABLE ERROR IN DELTA = 1.0 EL.DEG.....

THE NECESSARY VALUE OF H WAS 0.050
 THE LARGEST ERROR ENCOUNTERED WAS 0.755
 THE NUMBER OF FUNCTION ENTRIES WAS 84

JOHNSON-WARD METHOD--VARIOUS H--EQN. (5.12)

.....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

TIME, SEC	VALUES OF DELTA FOR VARIOUS H BEGINNING AT H = 0.200.				
0.000	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727
0.050			20.569	20.569	20.569
0.075				23.561	23.561
0.100		27.601	27.595	27.594	27.594
0.125				32.523	32.523
0.150			38.182	38.178	38.178
0.175				44.377	44.376
0.200	51.315	50.986	50.942	50.932	50.931
0.225				57.670	57.668
0.250			64.450	64.433	64.430
0.275				71.090	71.087
0.300		77.717	77.567	77.542	77.538
0.325				83.718	83.714
0.350			89.610	89.577	89.572
0.375				95.106	95.101
0.400	102.442	100.627	100.354	100.313	100.308
0.425				105.226	105.220
0.450			109.938	109.889	109.882
0.475				114.357	114.349
0.500		119.177	118.759	118.697	118.689
0.525				122.988	122.978
0.550			127.395	127.316	127.306
0.575				131.783	131.771
0.600	141.646	137.332	136.607	136.502	136.488
0.625				140.681	140.665
0.650			143.776	143.619	143.598
0.675				145.518	145.492
0.700		148.546	146.781	146.517	146.482
0.725				146.687	146.641
0.750			146.498	146.041	145.981
0.775				144.531	144.453
0.800	180.694	148.371	142.835	142.049	141.946
0.825				138.417	138.284
0.850			134.723	133.391	133.218
0.875				126.653	126.430
0.900		136.156	120.012	117.824	117.541
0.925				106.496	106.141
0.950			95.682	92.304	91.870
0.975				75.063	74.548
1.000	433.669	97.789	59.540	54.956	54.372

.....FOR ALLOWABLE ERROR IN DELTA = 1.0 EL.DEG.....

THE NECESSARY VALUE OF H WAS 0.025
 THE LARGEST ERROR ENCOUNTERED WAS 0.583
 THE NUMBER OF FUNCTION ENTRIES WAS 164

COLLATZ PROCEDURE--VARIOUS H--EQN. (5.17)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

TIME, SEC	VALUES OF DELTA FOR VARIOUS H BEGINNING AT H = 0.200.			
0.000	18.105	18.105	18.105	18.105
0.025				18.727
0.050			20.568	20.569
0.075				23.561
0.100		27.582	27.593	27.594
0.125				32.523
0.150			38.175	38.178
0.175				44.375
0.200	50.196	50.881	50.927	50.930
0.225				57.667
0.250			64.426	64.430
0.275				71.087
0.300		77.482	77.534	77.538
0.325				83.713
0.350			89.568	89.572
0.375				95.100
0.400	100.261	100.256	100.304	100.307
0.425				105.219
0.450			109.879	109.881
0.475				114.348
0.500		118.628	118.685	118.688
0.525				122.977
0.550			127.301	127.304
0.575				131.769
0.600	136.240	136.386	136.481	136.486
0.625				140.662
0.650			143.586	143.595
0.675				145.487
0.700		146.143	146.461	146.476
0.725				146.633
0.750			145.948	145.971
0.775				144.440
0.800	137.499	141.089	141.893	141.930
0.825				138.263
0.850			133.131	133.191
0.875				126.395
0.900		115.254	117.399	117.496
0.925				106.085
0.950			91.648	91.801
0.975				74.466
1.000	30.438	49.436	54.073	54.280

....FOR ALLOWABLE ERROR IN DELTA = 1.0 EL.DEG.....
 THE NECESSARY VALUE OF H WAS 0.050
 THE LARGEST ERROR ENCOUNTERED WAS 0.207
 THE NUMBER OF FUNCTION ENTRIES WAS 63

SECOND ORDER METHOD--VARIOUS H--EQN. (5.46)

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

TIME, SEC	VALUES OF DELTA FOR VARIOUS H BEGINNING AT H = 0.200.	
0.000	18.105	18.105
0.025		
0.050		
0.075		
0.100		27.591
0.125		
0.150		
0.175		
0.200	50.784	50.920
0.225		
0.250		
0.275		
0.300		77.528
0.325		
0.350		
0.375		
0.400	100.316	100.302
0.425		
0.450		
0.475		
0.500		118.687
0.525		
0.550		
0.575		
0.600	136.590	136.485
0.625		
0.650		
0.675		
0.700		146.459
0.725		
0.750		
0.775		
0.800	141.798	141.892
0.825		
0.850		
0.875		
0.900		117.397
0.925		
0.950		
0.975		
1.000	53.011	53.984

....FOR ALLOWABLE ERROR IN DELTA = 1.0 EL.DEG.....
 THE NECESSARY VALUE OF H WAS 0.200
 THE LARGEST ERROR ENCOUNTERED WAS 0.973
 THE NUMBER OF FUNCTION ENTRIES WAS 20

DEVOGELAERE METHOD--VARIOUS H--EQN. A.1-.5

....EXAMPLE NO. 1, CLEARING TIME 0.60 SECONDS.

TIME, SEC	VALUES OF DELTA FOR VARIOUS H BEGINNING AT H = 0.200.				
0.000	18.105	18.105	18.105	18.105	18.105
0.025				18.727	18.727
0.050			20.568	20.569	20.569
0.075				23.561	23.561
0.100		27.569	27.593	27.594	27.594
0.125				32.523	32.523
0.150			38.175	38.178	38.178
0.175				44.375	44.375
0.200	49.463	50.856	50.926	50.930	50.931
0.225				57.667	57.668
0.250			64.425	64.430	64.430
0.275				71.087	71.087
0.300		77.442	77.533	77.538	77.538
0.325				83.713	83.713
0.350			89.567	89.571	89.572
0.375				95.100	95.100
0.400	98.010	100.212	100.303	100.307	100.307
0.425				105.219	105.220
0.450			109.878	109.881	109.881
0.475				114.348	114.348
0.500		118.579	118.684	118.688	118.688
0.525				122.977	122.977
0.550			127.300	127.304	127.304
0.575				131.769	131.769
0.600	132.020	136.317	136.480	136.485	136.486
0.625				140.660	140.662
0.650			143.556	143.589	143.594
0.675				145.478	145.486
0.700		145.550	146.363	146.462	146.475
0.725				146.613	146.632
0.750			145.752	145.944	145.968
0.775				144.404	144.437
0.800	107.157	138.606	141.542	141.881	141.925
0.825				138.199	138.256
0.850			132.526	133.108	133.182
0.875				126.288	126.384
0.900		108.221	116.399	117.359	117.482
0.925				105.913	106.067
0.950			90.106	91.590	91.780
0.975				74.215	74.441
1.000	-37.090	36.058	52.010	53.995	54.251

....FOR ALLOWABLE ERROR IN DELTA = 1.0 EL.DEG.....

THE NECESSARY VALUE OF H WAS 0.025

THE LARGEST ERROR ENCOUNTERED WAS 0.256

THE NUMBER OF FUNCTION ENTRIES WAS 85

BIBLIOGRAPHY

1. Kimbark, E.W., "Power System Stability", Vol. 1, John Wiley, New York, N.Y., 1948, p. 30 - 50.
2. Johnson, D.L., and Ward, J.B., "The Solution of Power Systems Stability Problems by Means of Digital Computers", A.I.E.E. Trans. 1957, 77, pt. 3, p.1321.
3. Lane, Long, and Powers, "Transient Stability Studies II - Automatic Digital Computation", *ibid.*, 1958, 77, pt. 3, p. 1291.
4. Stagg, Gabrielle, and Hohenstein, "Calculation of Transient Stability Problems Using a High-Speed Digital Computer", *ibid.*, 1959, 78, pt. 3, p. 566.
5. Rao, N. Dharma, "A New Approach to the Transient Stability Problem", Trans. A.I.E.E., 1962, pt. 3, June issue, p. 186.
6. Rao, H.R. and Rao, N.D., "Solution of Transient Stability Problems Through the Number Series Approach", Proc. I.E.E., London, Vol. III, No. 4, April, 1964, p. 775.
7. Levy, H. and Baggott, E.A., "Numerical Solutions of Differential Equations" (Book), Dover, New York, N.Y., 1950.
8. Collatz, L., "Numerische Behandlung von Differential Gleichungen", (book), Springer-Verlag, Berlin, 1951.
9. Kutta, W., "Beitrag zur Naeherungsweise Integration Totales Differentialgleichungen", Z. Math. Phys., Vol. 46, 1901, p. 435-453.
10. Gill, S., "A Process for the Step-by-Step Integration of Differential Equations in an Automatic Digital Computing Machine", Proc. Camb. Philos. Soc., Vol. 47, 1951, p. 96-108.
11. Martin, D.W., "Runge-Kutta Methods for Integrating Differential Equations on High-Speed Digital Computers", Comp. J., Vol. 1, 1958, p. 118-123.

12. Lance, G.N., "Numerical Methods for High-Speed Computers", (book), Iliffe and Sons, London, 1960.
13. McCracken and Dorn, "Numerical Methods and Fortran Programming", John Wiley, New York, N.Y. (book), 1964.
14. De Vogelaere, R., "A Method for the Numerical Integration of Differential Equations of Second Order without Explicit First Derivatives", J. Res. Nat. Bur. Stand., 54, p. 119-125.
15. Notes on App. Sci. No. 16, "Modern Computing Methods", H.M.S.O. London, 1961, p. 87-88.
16. Hildebrand, F.B., "Introduction to Numerical Analysis", McGraw-Hill, New York, N.Y. (book), 1956, p. 237-238.

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